CO-ORDINATE GEOMETRY

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U. N. DHUR & SONS, LTD

BOOKSELLERS & PUBLISHERS

15, BANKIM CHATTERJEE ST., CALCUTTA

Published by
UPENDRANATH DHUR,
For U. N. DHUR & SONS, LTD.
15, Bankim Chatterjee St., Calcutta

Printed by
TRIDIBESH BASU, B. A.
THE K. P. BASU PRINTING WORKS:
11, Mohendra Gossain Lane, Calcutta.

PREFACE

THE present treatise on "*Co-ordinate Geometry*" aims at presenting, within a short compass, the cardinal principles of the subject and their more important applications and illustrations, so essential to a proper understanding of the subject. We submit that the special features of the book are preference shown to elementary or *à priori* reasoning (whenever possible), exclusion of irrelevant details and inclusion of those select topics, which have a direct bearing on the University curriculum for a graduate course. In a word, every attempt has been made to meet the requirements of candidates, who go in for the B.A. and B.SC. (Pass) examinations of Indian Universities. Perhaps there are certain features in the book which have some interest for a wider class of '*casual readers*', who want to pick up a working knowledge of the subject in the shortest possible time. It is for the educated community to judge how far we have been successful in our enterprise.

In conclusion we tender our cordial thanks to Messrs. U. N. Dhur & Sons, Ltd., for the commendable promptitude with which they have undertaken the publication of the book. Our thanks are also due to the authorities and staff of Messrs. K. P. Basu Printing Works for the efficient discharge of their duties, in spite of their various preoccupations. We shall be grateful to any one, who will point out errors or misprints or offer suggestions for improvement.

CALCUTTA, *June, 1946* — HARIDAS BAGCHI, BHOLANATH MUKHERJEE

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CO-ORDINATE GEOMETRY

CHAPTER I

INTRODUCTION

1·1. We shall give at the outset a few important algebraical theorems and results, which though not properly belonging to the subject matter of Co-ordinate Geometry, are of frequent occurrence in the treatment of the subject and with which the students are not already familiar.

1·2. Determinant of the Second Order.

The symbolic notation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is called a *determinant of the second order* and stands for the expression $a_1b_2 - a_2b_1$, which is called its expansion. The letters a_1, a_2, b_1, b_2 are called the constituents of the determinant. *Constituents* arranged horizontally form a *row*, and those arranged vertically form a *column*. In the above determinant there are two rows and two columns, and the *number of constituents* is equal to 2×2, *i.e.*, 4.

1·3. Properties.

Theo. 1. *The value of the determinant is not altered by changing the rows into columns or columns into rows.*

Thus, $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Theo. 2. *If two rows or two columns of the determinant are interchanged, the sign of the determinant is changed.*

Thus, $$\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = b_1a_2 - a_1b_2 = -(a_1b_2 - a_2b_1) = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

1·4. Determinant of the Third Order.

The symbolic notation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \cdots \qquad \cdots \quad (1)$$

is called a *determinant of the third order* and stands for the expression

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad (2)$$

i.e., $a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$ (3)

$$\text{or } a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad (4)$$

i.e., $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2)$
$+ c_1(a_2b_3 - a_3b_2)$. ··· (5).

The results (3) and (4) are obviously identical. A determinant of the 3rd order is therefore reduced to a sum of three determinants of the 2nd order by the following rule :

Multiply each constituent of any row (or column) by the determinant obtained by omitting the row and column containing the constituent. Prefix the + and − signs alternately to the products thus obtained and add the results algebraically.

In a determinant of the 3rd order, there are 3 rows and 3 columns and the number of constituents is equal to 3×3 *i.e.*, 9.

1·5. Co-factors.

Determinants are very often shortly denoted by D or Δ. For the sake of brevity and convenience, the coefficients of $a_1, b_1, c_1, a_2, b_2, c_2, \ldots$ in the expansion of the determinant are very often denoted by the corresponding capital letters $A_1, B_1, C_1, A_2, B_2, C_2, \ldots$, which are called *Co-factors.*

Thus denoting the above determinant by D, the relation (3) can be written as*

$$D = a_1A_1 + a_2A_2 + a_3A_3, \qquad \cdots \quad (6)$$

where $A_1 = b_2c_3 - b_3c_2$, $A_2 = b_3c_1 - b_1c_3$,

$A_3 = b_1c_2 - b_2c_1$.

Similarly (4) can be written as

$$D = a_1A_1 + b_1B_1 + c_1C_1. \qquad \cdots \quad (7)$$

Similarly it can be shown that

$$D = b_1B_1 + b_2B_2 + b_3B_3$$
$$= c_1C_1 + c_2C_2 + c_3C_3, \text{ etc.}$$

Thus, *the sum of the products obtained by multiplying the constituents of any row (or column) by the corresponding co-factors is equal to the value of the determinant.*

Again $\quad b_1A_1 + b_2A_2 + b_3A_3$

$$= b_1(b_2c_3 - b_3c_2) + b_2(b_3c_1 - b_1c_3) + b_3(b_1c_2 - b_2c_1)$$
$$= 0.$$

Similarly, $a_2A_1 + b_2B_1 + c_2C_1 = 0$, etc.

Thus, *the sum of the products obtained by multiplying the constituents of any row (or column) by the corresponding co-factors of the constituents of any other row (or column) is zero.*

1·6. Properties.

Theo. 1. *The value of the determinant is not altered by changing the rows into columns or columns into rows.*

Thus, $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

This can easily be verified by expansion as in the case of the 2nd order determinant.

Theo. 2. *If two rows (or columns) are interchanged, the sign of the determinant is changed.*

Thus, $$\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This can be verified by expansion.

Theo. 3. *If two rows (or columns) of the determinant are identical, the value of the determinant is zero.*

Thus, $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Theo. 4. *If all the constituents of a row (or column) be multiplied by the same quantity, the determinant is multiplied by the same quantity.*

Thus, $$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Expanding in terms of the constituents of the first row, the left side $= \Sigma ma_1(b_2c_3 - b_3c_2) = m\Sigma a_1(b_2c_3 - b_3c_2) = mD$.

Theo. 5. *If each constituent of any row (or column) is given as the sum of two numbers, the determinant can be expressed as the sum of two determinants whose remaining rows (or columns) are unaltered.*

Thus,

$$\begin{vmatrix} a_1+x_1 & b_1+y_1 & c_1+z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Expanding in terms of the constituents of the first row,

the left side $= \Sigma(a_1 + x_1)(b_2c_3 - b_3c_2)$

$= \Sigma a_1(b_2c_3 - b_3c_2) + \Sigma x_1(b_3c_3 - b_3c_2)$

$= D_1 + D_2$, where D_1, D_2 are the two determinants on the right side.

1·7. The Determinant

$$\begin{vmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{h} & \mathbf{b} & \mathbf{f} \\ \mathbf{g} & \mathbf{f} & \mathbf{c} \end{vmatrix}.$$

The above determinant plays a very important part in Co-ordinate Geometry. Denoting the determinant by D, we have

$$D = a(bc - f^2) - h(hc - gf) + g(fh - bg)$$
$$= \mathbf{abc + 2fgh - af^2 - bg^2 - ch^2}.$$

Here $A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$,
$F = gh - af$, $G = hf - bg$, $H = fg - ch$.

It can be easily verified that

$$BC - F^2 = aD, \quad CA - G^2 = bD, \quad AB - H^2 = cD,$$
$$GH - AF = fD, \; HF - BG = gD, \; FG - CH = hD.$$

Thus, $BC - F^2 = (ca - g^2)(ab - h^2) - (gh - af)^2$

$= a(abc + 2fgh - af^2 - bg^2 - ch^2)$

$= aD.$

Similarly for others.

Hence, when $D=0$,

$$BC=F^2, \quad CA=G^2, \quad AB=H^2,$$
$$GH=AF, \quad HF=BG, \quad FG=CH.$$

1·8. Determinant of the Fourth order.

The symbolic notation

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

is called a *determinant of the fourth order.* It can be expanded in terms of the constituents of the first row or column by a rule similar to that used in expanding the determinant of the third order. Thus the above determinant is equivalent to

$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

or
$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

Properties established for a determinant of the 3rd order in Art. 1·6 are equally true for that of a 4th order.

1·9. Elimination.

Suppose we are required to eliminate x and y from the three equations

$$a_1x+b_1y+c_1=0, \quad \dots \quad (1)$$
$$a_2x+b_2y+c_2=0, \quad \dots \quad (2)$$
$$a_3x+b_3y+c_3=0. \quad \dots \quad (3)$$

From (2) and (3), we have by the rule of cross-multiplication

$$\frac{x}{b_2c_3-b_3c_2}=\frac{y}{c_2a_3-c_3a_2}=\frac{1}{a_2b_3-a_3b_2}.$$

$$\therefore \quad x=\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}, \quad y=\frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}.$$

Substituting these values of x, y in (1) and simplifying, we get

$$a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0,$$

i.e.,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0, \quad \dots \quad (4)$$

which is the required *eliminant.*

Similarly, the result of eliminating x, y, z from the equations

$$a_1x+b_1y+c_1z=0 \quad \dots \quad (5)$$
$$a_2x+b_2y+c_2z=0 \quad \dots \quad (6)$$
$$a_3x+b_3y+c_3z=0 \quad \dots \quad (7)$$

would be the same as (4). This can be easily shown by obtaining the proportional values of x, y, z from (6) and (7) by the rule of cross-multiplication and then substituting them in (5). In fact, in this case we are really eliminating two unknown quantities x/z, y/z from three equations.

Thus, *the result of elimination* of two quantities or three quantities (as in the second case) from three equations *is*

the vanishing of the determinant formed by the coefficients of the given equations, taken in the order in which they occur.

Note. The relation (4) is the condition which must hold among the coefficients of the given equations in order that x, y, z may have the same values, different from zero, in each of the three given equations.

Similarly, the result of elimination of three quantities x, y, z from four equations $a_1x+b_1y+c_1z+d_1=0$, $a_2x+b_2y+c_2z+d_2=0$, $a_3x+b_3y+c_3z+d_3=0$, $a_4x+b_4y+c_4z+d_4=0$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}=0.$$

1·10. Relations between roots and coefficients.

If α, β, γ be the roots of the cubic equation

$$ax^3+bx^2+cx+d=0,$$

then

$$\alpha+\beta+\gamma=-b/a,$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=c/a,$$

$$\alpha\beta\gamma=-d/a.$$

Since α, β, γ are the roots of the cubic,

$$\therefore\quad ax^3+bx^2+cx+d=a(x-\alpha)(x-\beta)(x-\gamma)$$

$$=a\{x^3-\Sigma\alpha.x^2+\Sigma\beta\gamma.x-\alpha\beta\gamma\}.$$

Hence comparing coefficients of x^2, x and constant terms, the required result follows.

Similar relations hold good between the roots and coefficients of equations of higher order.

Examples I

1. Find the value of

$$\text{(i)}\ \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}; \qquad \text{(ii)}\ \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix};$$

$$\text{(iii)}\ \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 14 & 21 \end{vmatrix}; \qquad \text{(iv)}\ \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ \frac{1}{bc} & \frac{1}{ca} & \frac{1}{ab} \end{vmatrix}.$$

2. Eliminate α, β from

(i) $a\alpha + b\beta + c = 0,$
$b\alpha + c\beta + a = 0,$
$c\alpha + a\beta + b = 0.$

(ii) $a_1\alpha + b_1\beta + c_1 = 0,$
$a_2\alpha + b_2\beta + c_2 = 0,$
$a_3\alpha + b_3\beta + c_3 = 0.$

3. Eliminate A, B, C from

(i) $Ax_1 + By_1 + C = 0,$
$Ax_2 + By_2 + C = 0,$
$Ax_3 + By_3 + C = 0.$

(ii) $Ax + By + C = 0,$
$Ax' + By' + C = 0,$
$Aa + Bb = 0.$

4. If α, β, γ be the roots of the cubic

(1) $2x^3 - 4x^2 + 5x - 6 = 0$, find the value of $\alpha\beta\gamma$.

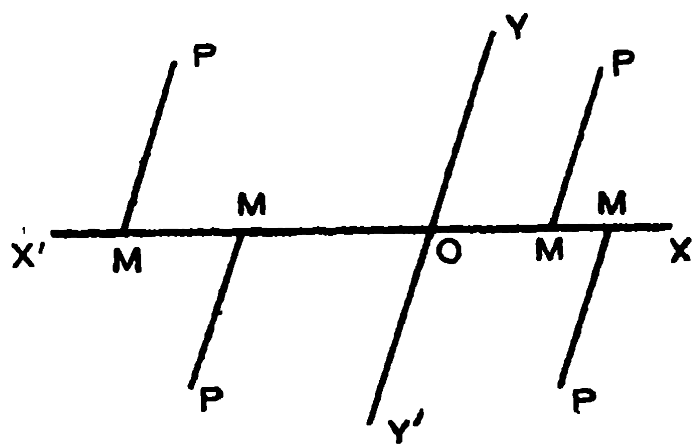
(2) $x^3 - 3x + 2 = 0$, find the value of $\Sigma\alpha$.

CHAPTER II

CO-ORDINATES

2·1. Cartesian Co-ordinates.

The underlying principle of Co-ordinate Geometry—first conceived by the French Mathematician Des Cartes—is to set up a definite correspondence between the position of a point in a plane and a pair of algebraic quantities, very often called *co-ordinates.*



The position of a point in a plane is referred to a pair of intersecting lines in the plane. Take any point O in the plane and an arbitrary pair of lines XOX' and YOY' through it. Let P be any point in the plane. Draw PM parallel to YOY' to meet XOX' in M. The two distances OM and MP which define the position of the point P in the plane are called its *co-ordinates.* The lines XOX', YOY' with reference to which the distances are measured are called axes of co-ordinates, XOX' being called the axis of x or x-axis and YOY', the axis of y or y-axis. The point O is called origin. OM is called the *x-co-ordinate* or *abscissa* and PM is called the *y-co-ordinate* or *ordinate* of P. If x be the abscissa and y the ordinate of P, P is very often denoted as the point (x, y). The co-ordinates x, y are often called *Cartesian* after the name of the discoverer,

Des Cartes, to distinguish them from other kinds of co-ordinates, which are also used to locate the position of a point in the plane.

2·2. Signs of Cartesian co-ordinates.

Even when the distances OM, MP are known, the position of P is not completely fixed, unless it is known whether OM is drawn to the right or left of YOY' and MP is drawn above or below XOX'. The directions of these lines are expressed by the signs of the co-ordinates of P, the same convention about the signs of the distances being adopted as in the case of Trigonometry.

Thus OM is considered positive when drawn to the right of YOY', and negative when drawn to the left of YOY'; and MP is considered positive when drawn above XOX', and negative when drawn below XOX'. The axes of co-ordinates divide the entire plane into four regions, each of which is called a *quadrant*. Thus XOY, YOX', $X'OY'$, $Y'OX$ are respectively first, second, third and fourth quadrants. The signs of the co-ordinates (x, y) of a point P in the different quadrants may be shown as follows:

P	I	II	III	IV
x	+	−	−	+
y	+	+	−	−

This convention regarding the algebraic signs of x, y being adopted, it is easy to see that any given point P has attached to it two determinate co-ordinates x and y and, when x and y are known both in magnitude and sign, the point P is uniquely determined.

2·3. Rectangular and oblique co-ordinates.

The co-ordinates as well as axes are said to be rectangular or oblique according as the angle XOY between the lines

of reference *i.e.*, axes is or is not a right angle. When the angle XOY is not a right angle, it is usually denoted by ω which may be greater or less than a right angle. The *co-ordinates used in this treatise are* for the most part Cartesian rectangular co-ordinates ; hence, wherever the term 'co-ordinates' occurs in the treatise, they may always be taken to mean *Cartesian and rectangular, unless the contrary is stated.*

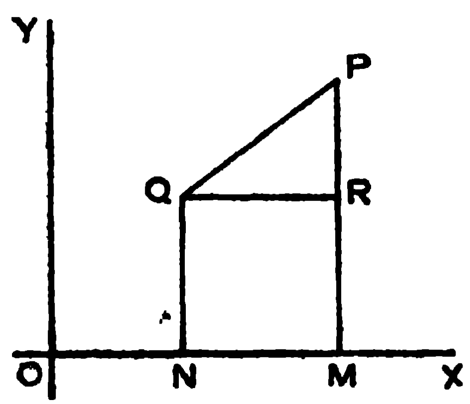
Cor. (i) Co-ordinates of the origin are (0, 0).

(ii) Co-ordinates of a point on the x-axis may be written as $(x, 0)$ and those on the y-axis as $(0, y)$.

2·4. Distance between two points.

(A) *Rectangular axes.*

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. Draw PM, QN perps. to OX. Draw $QR \parallel$ to OX to meet PM in R.



Then from the right-angled $\triangle PQR$, we have

$$PQ^2 = QR^2 + RP^2.$$

Now, $QR = NM = OM - ON$

$$= x_1 - x_2,$$

$$RP = MP - MR$$

$$= MP - NQ = y_1 - y_2.$$

$$\therefore \; PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

$\therefore$ PQ, *i.e.*, the distance between two points (x_1, y_1), (x_2, y_2)

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \qquad \ldots \quad (1)$$

Cor. The distance of the point (x_1, y_1) from the origin

$$= \sqrt{x_1^2 + y_1^2}.$$

(B) *Oblique axes.*

Draw PM, QN $\parallel$ to OY and QR $\parallel$ to OX.

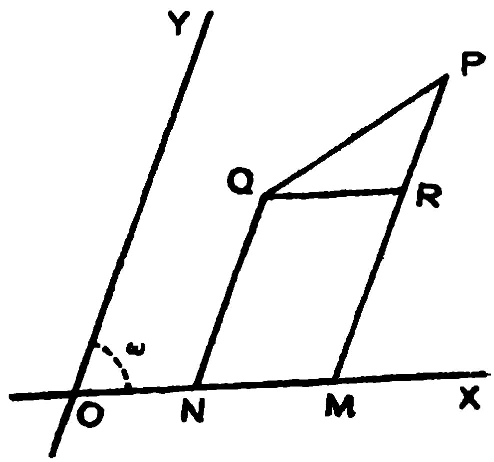
As before, $QR = x_1 - x_2$,

$RP = y_1 - y_2$.

Also $\angle QRP = \pi - \omega$.

$\therefore$ $\cos QRP = -\cos \omega$.

Then from $\triangle PQR$,



$$PQ^2 = QR^2 + RP^2 - 2QR.RP \cos QRP$$
$$= (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2) \cos \omega. \quad \ldots \quad (2)$$

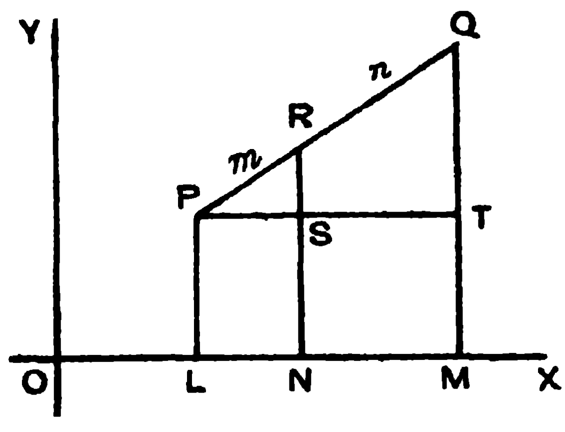
2·5. Section of a line in a given ratio.

To find the co-ordinates of a point which divides the line joining the points (x_1, y_1), (x_2, y_2) in the ratio $m : n$.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points and let $R(x, y)$ be the point dividing PQ in the ratio $m : n$, *i.e.*,

$$PR : RQ = m : n.$$

(i) First suppose R divides PQ internally, so that R lies between P and Q.



Draw PL, QM, RN $\parallel$ to OY. Draw PT $\parallel$ to OX, cutting RN in S and QM in T. Then since the parallels PL, QM, RN are met by the transversals PQ and LM,

$$\therefore \quad \frac{LN}{NM} = \frac{PR}{RQ} = \frac{m}{n}.$$

$$\therefore \quad \frac{ON-OL}{OM-ON}, \quad i.e., \quad \frac{x-x_1}{x_2-x}=\frac{m}{n}.$$

$$\therefore \quad x=\frac{mx_2+nx_1}{m+n}.$$

Again, in $\triangle\, PQT$, since RS is $\parallel$ to QT,

$$\therefore \quad \frac{RS}{QT}=\frac{PR}{PQ}=\frac{PR}{PR+RQ}=\frac{m}{m+n}.$$

$$\therefore \quad \frac{RN-SN}{QM-TM}, \quad i.e., \quad \frac{y-y_1}{y_2-y_1}=\frac{m}{m+n}.$$

$$\therefore \quad y=\frac{my_2+ny_1}{m+n}.$$

Hence the required co-ordinates of R are

$$\frac{\mathbf{mx_2+nx_1}}{\mathbf{m+n}}, \ \frac{\mathbf{my_2+ny_1}}{\mathbf{m+n}}.$$

(ii) Secondly, when R divides PQ *externally* in the same ratio $m : n$, *i.e.*, when R lies on PQ or QP produced, we can similarly prove by drawing the requisite figure that the co-ordinates of R are

$$\frac{\mathbf{mx_2-nx_1}}{\mathbf{m-n}}, \ \frac{\mathbf{my_2-ny_1}}{\mathbf{m-n}}.$$

Cor. Co ordinates of the mid-point of PQ are $\frac{1}{2}(\mathbf{x_1+x_2})$, $\frac{1}{2}(\mathbf{y_1+y_2})$.

Note 1. It is easy to see that *the above results are equally true for oblique axes.*

Note 2. From above it is clear that the co-ordinates of a point dividing PQ in the ratio $\lambda : 1$ are

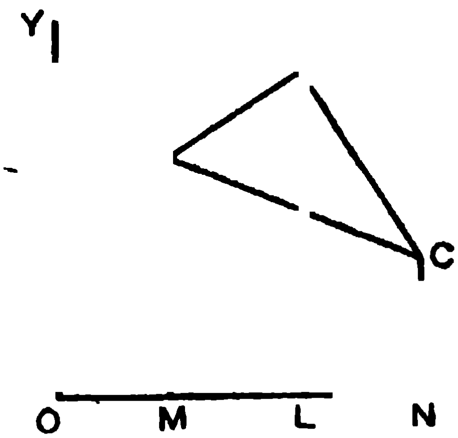
$$\frac{x_1+\lambda x_2}{1+\lambda}, \ \frac{y_1+\lambda y_2}{1+\lambda}.$$

Considering λ as a variable parameter (*i.e.*, capable of all values positive or negative) *the co-ordinates of any point on the join of* (x_1, y_1) *and* (x_2, y_2) *can be represented in the above forms.*

2·6. Area of a triangle.

To find the area of a triangle in terms of the co-ordinates of its vertices.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the co-ordinates of the vertices A, B, C of $\Delta\ ABC$. Draw AL, BM, CN perp. to the x-axis and let Δ denote the area of the triangle. Then Δ = trapz. $ALMB$ + trapz. $ALNC$ − trapz. $BMNC$. Since area of a trapz. $= \frac{1}{2}$ (sum of parl. sides) × distance between them,



$$\therefore \quad \Delta = \tfrac{1}{2}(AL + BM) \times ML + \tfrac{1}{2}(AL + CN) \times LN - \tfrac{1}{2}(BM + CN) \times MN$$
$$= \tfrac{1}{2}(y_1 + y_2)(x_1 - x_2) + \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1) - \tfrac{1}{2}(y_2 + y_3)(x_3 - x_2).$$

On simplifying and re-arranging, we easily have

$$\Delta = \tfrac{1}{2}(\mathbf{x}_1\mathbf{y}_2 - \mathbf{x}_2\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_3 - \mathbf{x}_3\mathbf{y}_2 + \mathbf{x}_3\mathbf{y}_1 - \mathbf{x}_1\mathbf{y}_3) \cdots (1)$$

which can be easily put in the determinant notation as

$$\Delta = \tfrac{1}{2}\begin{vmatrix} \mathbf{x}_1 & \mathbf{y}_1 & 1 \\ \mathbf{x}_2 & \mathbf{y}_2 & 1 \\ \mathbf{x}_3 & \mathbf{y}_3 & 1 \end{vmatrix}.$$

Cor. The area of the triangle whose vertices are (0, 0), (x_1, y_1), (x_2, y_2) is $\frac{1}{2}(x_1y_2 - x_2y_1)$.

Note 1. Obviously when 3 points are collinear, the area of the resulting triangle $\Delta = 0$. Conversely, if $\Delta = 0$, then the 3 vertices must be collinear. For the area being = $\frac{1}{2}$ base × altitude, it can vanish if either base or altitude = 0. Now, base cannot be zero, for that would imply 2 of the points to coincide. So the altitude, *i.e.*, the

perp. dropped from one of the vertices to the opposite base $=0$. In other words the 3 vertices must be collinear. Thus, *the vanishing of Δ is the necessary and sufficient condition for the collinearity of 3 points.* For another method see Art. 3·11.

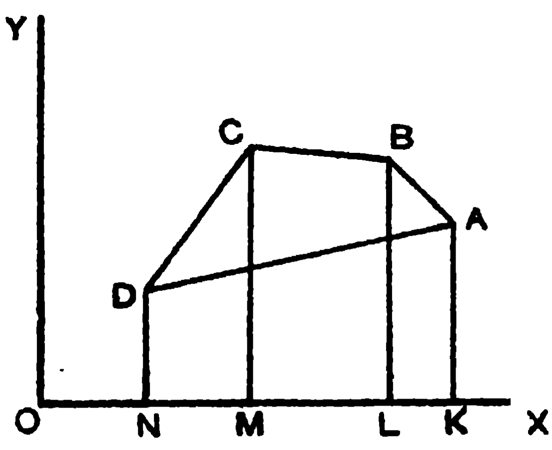
Note 2. If *the axis be oblique* and inclined at angle ω, the perps. AL, BM, CN are no longer equal to the ordinates y_1, y_2, y_3 but to $y_1 \sin\omega$, $y_2 \sin\omega$, $y_3 \sin w$. Hence in this case the area of ΔABC becomes

$$\tfrac{1}{2}\sin\omega\{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\}.$$

Note 3. Here we have taken the vertices of the triangle in the counter-clockwise order ; but if we take them in the clockwise order, the analytical expression for the area of the triangle, while retaining its magnitude, will be found to have changed its sign.

2·7. Area of a quadrilateral.

To find the area of a quadrilateral in terms of the co-ordinates of its angular points.



Let the angular points of the quadrilateral taken in order be $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$. Draw AK, BL, CM, DN perps. on OX.

Area of the quad. $ABCD$ = trap. $AKLB$ + trap. $BLMC$ + trap. $CMND$ − trap. $DNKA$

$$=\tfrac{1}{2}(AK+BL).KL+\tfrac{1}{2}(BL+CM).LM + \tfrac{1}{2}(CM+DN).MN-\tfrac{1}{2}(DN+AK).NK$$

$$=\tfrac{1}{2}(y_1+y_2)(x_1-x_2)+\tfrac{1}{2}(y_2+y_3)(x_2-x_3) + \tfrac{1}{2}(y_3+y_4)(x_3-x_4)-\tfrac{1}{2}(y_1+y_4)(x_1-x_4)$$

$$=\tfrac{1}{2}\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3) + (x_4y_1-x_1y_4)\},$$

on omitting the terms which cancel.

Note 1. The above formula can also be established by joining the origin O to the vertices A, B, C, D and thus dividing the quadrilateral into a number of triangles. Thus

Quad $ABCD = \Delta OAB + \Delta OBC - \Delta OCD - \Delta ODA$.

Now, $\Delta OAB = \frac{1}{2}(x_1 y_2 - x_2 y_1)$. [See Cor., Art. 2·6]

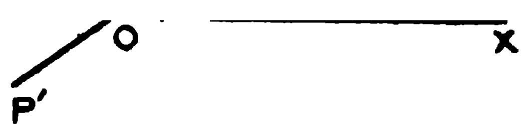
Similarly for other triangles.

The area of a polygon of any number of sides can similarly be obtained.

Note 2. It should be noted that the area found above has a reference to the certain order in which the vertices A, B, C, D are taken.

2·8. Polar Co-ordinates.

There is another way in which the position of a point in a plane can be fixed. Take a fixed point O and a fixed straight line OX through it. The position of any point P will be known, when we know the length of OP and the angle XOP. OP, usually denoted by r, is called the *radius vector* and $\angle XOP$, usually denoted by θ, is called the *vectorial angle*. The point O is called the *pole* and the line OX called the *initial line*.



Thus, (r, θ) which define the position of the point P are called *polar co-ordinates* of P.

As in Trigonometry, θ is considered *positive*, if measured from OX in the *counter-clockwise* direction and *negative*, if measured in the *clockwise* direction.

The radius vector r is considered to be positive, if measured from O along the line bounding the vectorial angle, and negative, if measured in the opposite direction.

Suppose PO is produced to P', so that OP' is equal in magnitude to OP. For the vectorial angle XOP *i.e.*, θ which OP bounds, OP is positive and OP' negative, but for the vectorial angle XOP' *i.e.*, $-(\pi - \theta)$ which OP'

bounds, OP' is positive and OP negative. Thus we see that the *same radius vector may be positive or negative according to the vectorial angle with which it is associated.* Since the same radius vector can be taken with two different signs and the vectorial angle can be measured in two different directions, *a point on a plane can* therefore *be represented in four ways by polar co-ordinates.* Thus the same point is represented by the following polar co-ordinates :

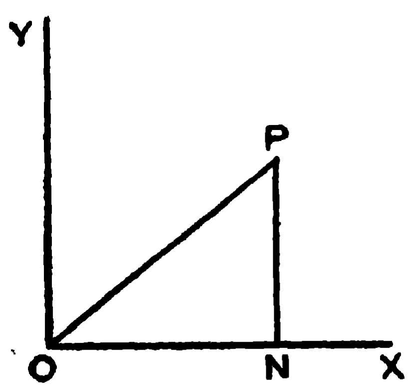
$$(r, \theta), (-r, \pi+\theta), \{-r, -(\pi-\theta)\}, \{r, -(2\pi-\theta)\}.$$

For isolated points, it is however always convenient to choose both r and θ positive.

Note. *Co-ordinates of the pole* are (0, 0).

2·9. Relation between Cartesian and Polar Co-ordinates.

Taking the pole and the initial line of the polar system as the origin and the axis of x in the Cartesian system, we can easily find out the relation between the co-ordinates of a point in the two systems.



Let (x, y) be the Cartesian co-ordinates of P with respect to OX, OY as axes and let (r, θ) be its polar co-ordinates with reference to O as pole and OX as initial line.

Draw PN perp. on OX and join OP.

Then, $\mathbf{x} = ON = OP \cos XOP = \mathbf{r} \cos \theta$. ... (1)

$\mathbf{y} = PN = OP \sin XOP = \mathbf{r} \sin \theta$. ... (2)

Squaring and adding (1) and (2), we get, $r^2 = x^2 + y^2$ and dividing (2) by (1), we get $\tan \theta = \dfrac{y}{x}$.

$$\therefore \quad \mathbf{r} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}. \quad \ldots \quad (3)$$

$$\theta = \tan^{-1} \frac{\mathbf{y}}{\mathbf{x}}. \quad \ldots \quad (4)$$

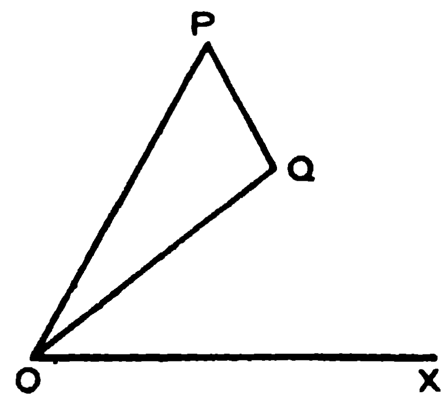
The relations (1) and (2) express Cartesian co-ordinates in terms of polar ones and (3) and (4) express polar equations in terms of Cartesian ones.

By means of the above two pairs of relations *the polar equation of a curve can be deduced from its Cartesian form* by writing $r\cos\theta$ for x and $r\sin\theta$ for y in the equation of the curve. Similarly, the *Cartesian equation of a curve can be deduced from its polar equation* by writing $\sqrt{x^2+y^2}$ for r and $\tan^{-1}y/x$ for θ in the equation of the curve.

Thus, if the Cartesian equation of a curve is $f(x, y)=0$, its polar equation is $f(r\cos\theta, r\sin\theta)=0$.

Conversely, if the polar equation of a curve is $F(r, \theta)=0$, its Cartesian equation is $F(\sqrt{x^2+y^2}, \tan^{-1}y/x)=0$.

2·10. Distance between two points in polar co-ordinates.



Let (r_1, θ_1), (r_2, θ_2) be the polar co-ordinates of P, Q.

Then $\angle POQ=\angle XOP-\angle XOQ=\theta_1-\theta_2$.

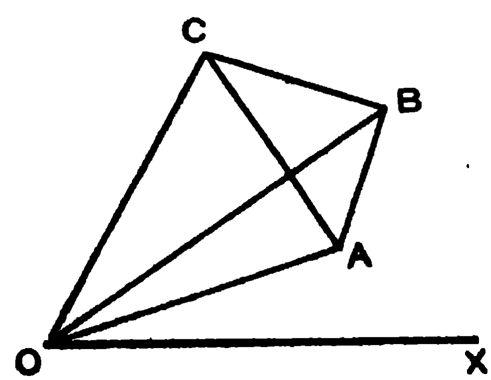
Hence PQ^2

$=OP^2+OQ^2-2OP.\,OQ\cos POQ$

$=r_1{}^2+r_2{}^2-2r_1r_2\cos(\theta_1-\theta_2)$.

2·11. Area of a triangle in polar co-ordinates.

To find the area of a triangle in terms of the polar co-ordinates of its vertices.



Let (r_1, θ_1), (r_2, θ_2), (r_3, θ_3) be the polar co-ordinates of the vertices A, B, C of the $\triangle ABC$. We have

$$\triangle ABC=\triangle OAB+\triangle OBC-\triangle OAC.$$

$$\triangle OAB=\tfrac{1}{2}OA.OB\sin AOB$$
$$=\tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1).$$

Similarly,

$\triangle OBC = \frac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2)$,

$\triangle OAC = \frac{1}{2} r_3 r_1 \sin(\theta_3 - \theta_1) = -\frac{1}{2} r_3 r_1 \sin(\theta_1 - \theta_3)$.

$$\therefore \quad \triangle = \tfrac{1}{2}[r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 - \theta_3) + r_1 r_2 \sin(\theta_2 - \theta_1)].$$

2·12. Equation and Locus.

We have seen that the position of a point in a plane is dependent upon two co-ordinates. When, however, a single analytical relation of the form $x + y - 7 = 0$, or $x^2 + y^2 = 4$ (1) is given, then, by giving various values to x, we get corresponding values of y. We can plot the points, whose co-ordinates are the corresponding values of x and y thus obtained. The points thus plotted will arrange themselves on a certain curve Σ, say. Then (1) is called the equation of the curve Σ and Σ is called the *graph* or the *locus* of the equation (1). What has been said with regard to the relation between the Cartesian co-ordinates applies equally well to the case of polar co-ordinates. Thus the **equation** of a curve is the algebraical relation, which is satisfied by the co-ordinates of every point on the curve and by no point outside it.

Conversely, the **locus** of an equation is the curve, the co-ordinates of every point on which satisfy that equation.

When a point moves in a curve, its co-ordinates are called *current* or *running* co-ordinates. It is usual to denote the current co-ordinates by (x, y), (r, θ) and the co-ordinates of a point in any of its particular positions by (x', y'), (x_1, y_1), (r_1, θ_1) etc. :

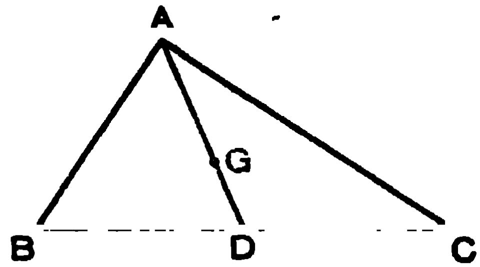
The equation of the locus of a moving point is obtained thus :

From the given geometrical condition of motion, find out a relation between the co-ordinates of the moving point in any of its positions and this relation, when expressed in current co-ordinates, is the equation of the locus. Sometimes when there is no chance of confusion, the co-ordinates of the moving point are taken in the current form in the very beginning.

2·13. Illustrative Examples.

Ex. 1. *If the vertices of a triangle are* (x_1, y_1), (x_2, y_2), (x_3, y_3), *show that the co-ordinates of its centroid are*

$$\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}$$



Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle. Let D be the mid-point of BC.

Then the coordinates of D are $\frac{1}{2}(x_2+x_3)$, $\frac{1}{2}(y_2+y_3)$.

Let $\bar{x}, \bar{y}$ be the co-ordinates of the centriod G. Since G divides AD in the ratio $2:1$,

$$\bar{x}=\frac{2.\frac{1}{2}(x_2+x_3)+x_1}{3}=\frac{x_1+x_2+x_3}{3},$$

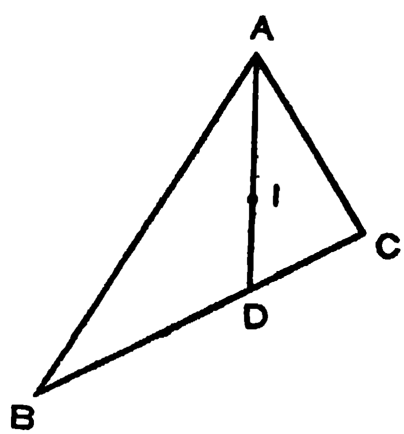
$$\bar{y}=\frac{2.\frac{1}{2}(y_2+y_3)+y_1}{3}=\frac{y_1+y_2+y_3}{3}.$$

Ex. 2. *If the vertices of a triangle are* (x_1, y_1), (x_2, y_2), (x_3, y_3), *show that co-ordinates of its in-centre are*

$$\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c},$$

where a, b, c are the sides of the triangle.

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle.



Let AD bisect $\angle BAC$ and cut BC in D.

Then we know that

$$BD : DC = BA : AC = c : b.$$

Then the co-ordinates of D are

$$\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}.$$

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the in-centre I.

Again, $ID : AD =$ in-radius : altitude

$$=\frac{\Delta}{s} : \frac{2\Delta}{a}^{*}$$

$$= a : a+b+c.$$

*See Das & Mukherjee's Higher Trigonometry, Art. 28.

$\therefore \qquad AI : ID = b+c : a$

$$\therefore \qquad \bar{x} = \frac{ax_1 + (b+c)\dfrac{bx_2+cx_3}{b+c}}{a+b+c} = \frac{ax_1+bx_2+cx_3}{a+b+c}.$$

Similarly $\bar{y}$ can be obtained.

Ex. 3. *Transform the equations*

(i) $(x^2+y^2)^2 = a^2(x^2-y^2)$

from Cartesian to polar co-ordinates ;

(ii) $r^2 \cos 2\theta = a^2$

from polar to Cartesian co-ordinates. [*C. U. 1941*]

(i) Putting $x = r\cos\theta$, $y = r\sin\theta$, we have $x^2+y^2=r^2$ and $x^2-y^2=r^2(\cos^2\theta-\sin^2\theta)=r^2\cos 2\theta$.

$\therefore$ the equation becomes

$$r^4 = a^2r^2\cos 2\theta,$$

$$i.e., \qquad r^2 = a^2\cos 2\theta.$$

(ii) The equation may be written as

$$r^2(\cos^2\theta - \sin^2\theta) = a^2,$$

$$i.e., \qquad x^2 - y^2 = a^2.$$

Ex. 4. *Find the equation of the locus of a point, which moves so that the sum of the squares of its distances from the points* $(a, 0)$, $(-a, 0)$ *is constant and equal to* $2c^2$.

Let (x', y') be the co-ordinates of the point in any of its position. Then, by the condition of the problem,

$$(x'-a)^2 + y'^2 + (x'+a)^2 + y'^2 = 2c^2,$$

$$i.e., \qquad x'^2 + y'^2 = c^2 - a^2.$$

This relation being satisfied by the co-ordinates of any point on the locus, the equation of the locus is $\quad x^2+y^2=c^2-a^2$.

Examples II

1. Show that the four points, whose co-ordinates are

$$(4, 3), (6, 4), (5, 6), (3, 5),$$

are the vertices of a square.

2. Find the circum-centre and circum-radius of the triangle, whose vertices are $(7, 9)$, $(1, 1)$, $(0, 2)$.

3. Prove that the three points, whose co-ordinates are
$$\{a \cos \alpha,\ a \sin \alpha\},\ \{a \cos (\alpha + \tfrac{2}{3}\pi),\ a \sin (\alpha + \tfrac{2}{3}\pi)\},$$
and $\{a \cos (\alpha - \tfrac{2}{3}\pi),\ a \sin (\alpha - \tfrac{2}{3}\pi)\}$ form an equilateral triangle, whose circumcentre is the origin.

4. Verify that the 4 points A, B, C, D, whose co-ordinates are (a, b), $(a+\alpha,\ b+\beta)$, $(a+\alpha+\alpha',\ b+\beta+\beta')$, and $(a+\alpha',\ b+\beta')$, when joined in order, form a parallelogram.

Also ascertain the condition for $ABCD$ to be

(i) a rhombus, (ii) a rectangle.

[*For* (i) *use* $AB = BC$. *For* (ii) *use* $AC = BD$.]

5. If (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) are the vertices of a parallelogram, taken in order, then
$$x_1 + x_3 = x_2 + x_4 \text{ and } y_1 + y_3 = y_2 + y_4.$$

6. If D be the mid-point of the side BC of $\triangle ABC$, then
$$AB^2 + AC^2 = 2(AD^2 + BD^2).$$

7. Prove analytically that the line joining the mid-points of the two sides of a triangle is half the third side.

8. Find the co-ordinates of the point, which divides the line joining the points $(a+2b,\ a-2b)$, $(a-2b,\ a+2b)$,

(i) internally and (ii) externally, in the ratio $a : b$.

9. (i) Satisfy yourselves that the two triangles, whose vertices are $(3, 0)$, $(0, 7)$, $(1, 1)$ and $(13, 3)$ $(2, 3)$, $(-11, 2)$ have the *same* area and the *same* centroid.

(ii) If two vertices of a triangle are $(2, 7)$ and $(6, 1)$ and its centroid is $(6, 4)$, find the third vertex.

10. If G be the centroid of $\triangle ABC$, then
$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

11. If P, Q be the points $(a \cos \varphi,\ b \sin \varphi)$, $(-a \sin \varphi,\ b \cos \varphi)$ and O be the origin, prove that the expression $OP^2 + OQ^2$ and the area of $\triangle OPQ$ are independent of φ.

12. Find the area of the triangle, whose vertices are

(i) $(3, 2), (5, 4), (2, 2)$;

(ii) $(a, a^2), (b, b^2), (c, c^2)$;

(iii) $(a \cos \varphi_1, b \sin \varphi_1), (a \cos \varphi_2, b \sin \varphi_2),$
$(a \cos \varphi_3, b \sin \varphi_3)$;

(iv) $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right), \left(ct_3, \frac{c}{t_3}\right).$

13. The angular points of two triangles are given as follow :

(i) $(at_1{}^2, 2at_1), (at_2{}^2, 2at_2), (at_3{}^2, 2at_3)$;

and (ii) $\{at_2t_3, a(t_2+t_3)\}, \{at_3t_1, a(t_3+t_1)\},$
$\{at_1t_2, a(t_1+t_2)\}.$

Show that the area of the first triangle is double that of the second.

14. Prove analytically that the area of any triangle is four times the area of the triangle, formed by joining the mid-points of its sides.

15. Verify that the area of the triangle formed by the points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3)$$

is the same as that of the triangle formed by the points

$$(x_1+h, y_1+k), (x_2+h, y_2+k), (x_3+h, y_3+k).$$

16. Find the area of the triangle, whose vertices are

$$(a, b+c), (b, c+a), (c, a+b)$$

and interpret the result geometrically.

✓**17.** Show that the three points $(4, 2), (7, 5), (9, 7)$ lie on a right line. [*C. U. 1943*]

18. Show that the lines, joining the middle points of the opposite sides of a quadrilateral and the line joining the middle points of its diagonals are concurrent and bisect one another.

19. Show that the four points whose co-ordinates are

$$a(\cos\alpha+\cos\beta+\cos\gamma),\quad a(\sin\alpha+\sin\beta+\sin\gamma),$$
$$a(\cos\beta+\cos\gamma+\cos\delta),\quad a(\sin\beta+\sin\gamma+\sin\delta),$$
$$a(\cos\gamma+\cos\delta+\cos\alpha),\quad a(\sin\gamma+\sin\delta+\sin\alpha),$$
$$a(\cos\delta+\cos\alpha+\cos\beta),\quad a(\sin\delta+\sin\alpha+\sin\beta)$$

lie on a circle.

[*Each point is equidistant from the point* $(a\Sigma\cos\alpha,\ a\Sigma\sin\alpha)$.]

20. (i) Find the area of the quadrilateral, whose angular points are $(1, 1), (3, 4), (5, -2), (4, -7)$. [*C. U. 1944*]

(ii) If the area of the quadrilateral whose angular points A, B, C, D, taken in order, are $(1, 2), (-5, 6), (7, -4)$ and $(k, -2)$ be zero, find the value of k. [*C. U. 1945*]

(iii) Show that the area of the quadrilateral, whose vertices, taken in order, are $(a, 0), (-b, 0), (0, a), (0, -b)$, is zero ; $(a, b > 0)$. Explain the result with the help of a diagram. [*C. U. 1943*]

21. Prove that the points $(3, 90°)$ and $(3, 30°)$ form with the origin an equilateral triangle.

22. Find the area of the triangle whose vertices are

(i) $(3, 30°), (2, 90°), (1, 150°)$.

(ii) $(a, \theta), (2a, \theta+\frac{1}{3}\pi), (3a, \theta+\frac{2}{3}\pi)$.

23. (*a*) Transform the following Cartesian equations into polar forms :

(i) $x^2+y^2-2ax=0$. (ii) $Ax+By+C=0$.

(iii) $y=x\tan\alpha$. (iv) $x^2+y^2=a^2$.

(*b*) Transform the following polar equations into Cartesian forms :

(i) $r=a$. (ii) $r=2a\cos\theta$.

(iii) $k/r=A\cos\theta+B\sin\theta$. (iv) $r^2=a^2\cos 2\theta$.

(v) $r^2\sin 2\theta=2a^2$. (vi) $r^{\frac{1}{2}}\cos\frac{1}{2}\theta=a^{\frac{1}{2}}$.

✓(vii) $r(\cos 3\theta+\sin 3\theta)=5k\sin\theta\cos\theta$.

[*C. U. 1931, '46*]

24. Find the equation of the locus of a point, which moves so that

(i) its distance from the origin is constant and equal to a.

(ii) the square of its distance from the x-axis is $4a$ times its distance from the y-axis.

(iii) its distance from the y-axis is twice its distance from the point (2, 2). [*C. U. 1939*]

(iv) the area of the triangle, formed by joining it to the points $(a, 0)$, $(-a, 0)$ and by joining these points, is constant and equal in magnitude to c^2.

(v) its distance from a fixed point (r_1, θ_1) is always constant and equal to a.

CHAPTER III

STRAIGHT LINE

3·1. Locus of $Ax+By+C=0$.

Every first degree equation in x and y always represents a straight line and conversely *every straight line can be represented by a first degree equation in x and y.*

Let us proceed to characterise the curve Γ defined by the linear Cartesian equation $Ax+By+C=0$. $\cdots$ (1)

If $P\ (x_1, y_1)$, $Q\ (x_2, y_2)$, $R\ (x_3, y_3)$ be three points taken at random on the curve Γ, we must have

$Ax_1+By_1+C=0$, $Ax_2+By_2+C=0$, $Ax_3+By_3+C=0$.

Eliminating A, B, C, we get

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$$

Hence, by Note 1 of Art. 2·6, P, Q, R are collinear, *i.e.*, any three points on Γ lie on a right line. So if two of the points, say P, Q be kept fixed, any third point R on Γ must lie on the line PQ. This is another way of saying that Γ itself is a right line. Thus every first degree equation in x and y always represents a straight line.

To prove the *converse* case, we start with any given straight line Γ and take any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on it. Regarding those points as *fixed* and taking an arbitrary point (x, y) on Γ, we immediately express the condition of collinearity in the form :

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0.$$

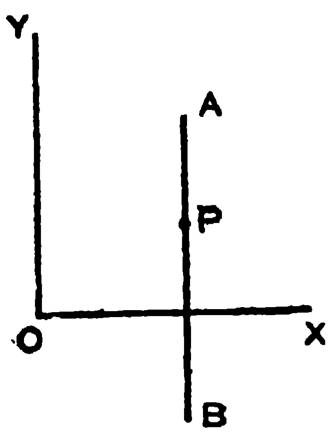
Expanding in terms of the constituents of the first row, it becomes $Ax + By + C = 0,$

where $A = y_1 - y_2,\ B = x_2 - x_1,\ C = x_1 y_2 - x_2 y_1.$

Because P, Q are *fixed* points, A, B, C must be regarded as constants, independent of the current co-ordinates x, y. Thus we have established the converse proposition that every straight line can be represented by a linear equation.

3·2. Equation of lines parallel to axes.

Let AB be a line parallel to y-axis and at a distance a from it. Let (x, y) be the co-ordinates of *any point* P on it.



Then $x = a$, whatever be the value of a.

$\therefore$ *the equation of a line parallel to y-axis* and at a distance a from it is $x = a$.

Similarly *the equation of a line parallel to x-axis* and at a distance b from it is $y = b$.

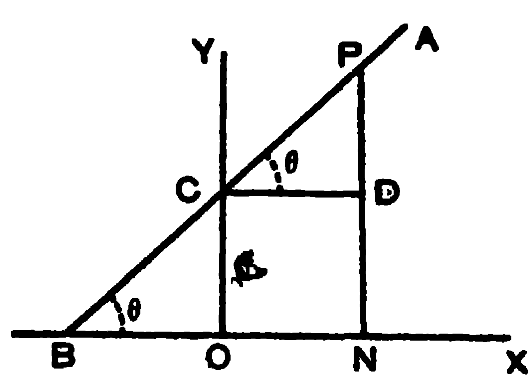
Cor. The *equation to the x-axis* is $y = 0$.

The *equation to the y-axis* is $x = 0$.

3·3. Equation of a straight line.

(A) **m-form.**

To find the equation to a straight line which cuts off a given intercept on the y-axis and is inclined at a given angle to the x-axis.



Let AB be the straight line, cutting off the intercept $OC = c$ from the y-axis and being inclined at an angle θ to the x-axis. Let (x, y) be the co-ordinates of any point P

on AB. Draw PN perp. to *x-axis* and CD parl. to y-axis to meet PN at D.

Then, $\tan\theta = \tan PCD = \frac{PD}{CD} = \frac{PN - CO}{ON} = \frac{y-c}{x}$.

$\therefore \quad y = x\tan\theta + c$

or $\quad \mathbf{y = mx + c} \qquad \cdots \qquad \cdots \quad (1)$

where $\quad m = \tan\theta$.

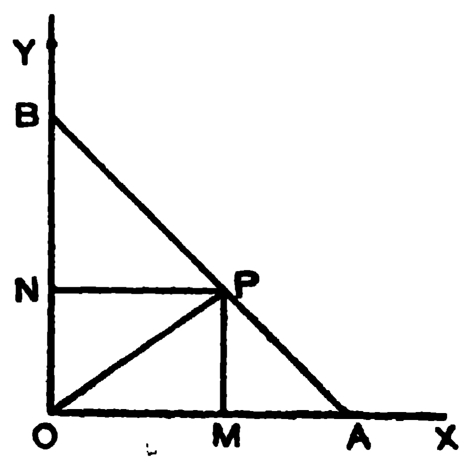
This, being the relation between the co-ordinates of any point on the line, is the equation of the line.

Note. The angle θ is called *slope* or *gradient* of the line.

(B) **Intercept-form.**

To find the equation to a straight line which cuts off given intercepts from the axis.

Let the straight line AB cut off the intercepts $OA = a$ and $OB = b$ from the axes of x and y respectively. Let (x, y) be the co-ordinates of any point P on AB. Draw PM, PN perps. to x-axis and y-axis respectively. Join OP. From the fig.,



$\triangle OPA + \triangle OPB = \triangle OAB.$

$\therefore \quad \frac{1}{2}OA.\ PM + \frac{1}{2}OB.\ PN = \frac{1}{2}OA.OB,$

i.e., $\frac{1}{2}ay + \frac{1}{2}bx = \frac{1}{2}ab$.

Dividing by $\frac{1}{2}ab$, we get

$$\mathbf{\frac{x}{a} + \frac{y}{b} = 1}. \qquad (2)$$

Note 1. ***The above equation also holds good for oblique axes ;*** **the only difference in the proof is that the factor sin ω, which occurs on both sides for the expression for the areas, cancels out finally.**

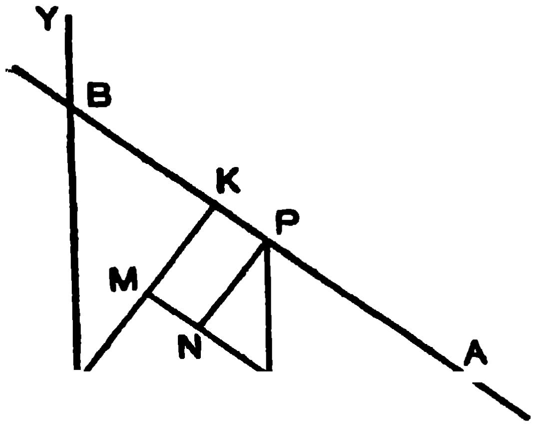
Note 2. The above equation can also be written as

$$\mathbf{lx + my = 1}$$

where $l = 1/a$, $m = 1/b$.

(C) **Perpendicular-form.**

To find the equation to a straight line in terms of the perpendicular p let fall on it from the origin and the angle α that this perpendicular makes with the axis of x.



Let OK be $\perp$ to the line AB from O and $\angle XOK = \alpha$. Let (x, y) be the co-ordinates of any point P on AB.

Draw PL perp. to OX, LM perp. to OK and PN perp. to LM.

Then, $\qquad OM = OL \cos\alpha = x\cos\alpha,$

$$MK = NP = LP \sin PLN = y \sin\alpha,$$

for $\quad \angle PLN = 90° - \angle MLO = \angle LOM = \alpha.$

Now $\qquad p = OK = OM + MK = x\cos\alpha + y\sin\alpha.$

$\therefore$ the reqd. equation of the line is

$$\mathbf{x \cos \alpha + y \sin \alpha = p.} \qquad \ldots \quad (3)$$

Alternative Method.

Since $\qquad OA = p/\cos\alpha,$

$$OB = p/\sin\alpha,$$

$\therefore$ using the intercept-form, the equation of the line is

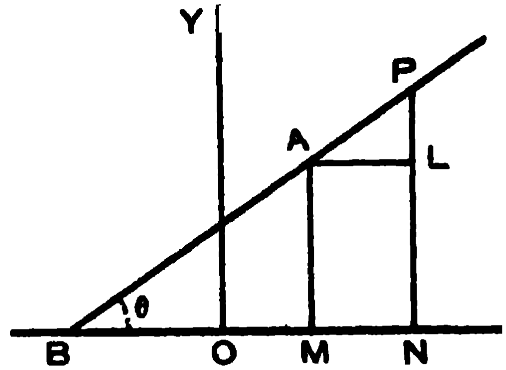
$$\frac{x}{p/\cos\alpha} + \frac{y}{p/\sin\alpha} = 1,$$

or, $\qquad x\cos\alpha + y\sin\alpha = p.$

(D) **Distance-form.**

To find the equation to a straight line, passing through a given point (x_1, y_1) and inclined at an angle θ to the x-axis.

Let A be the given point (x_1, y_1) and AB be the straight line through A inclined at an angle θ with x-axis.



Then $\angle PBX = \theta$.

Let (x, y) be the co-ordinates of any point P on the line, where $AP = r$. Draw AM, PN perps. to OX, and AL perp. to PN. Then $\angle PAL = \theta$,

$$AL = MN = ON - OM = x - x_1,$$

$$PL = PN - NL = PN - AM = y - y_1.$$

Now, $AL = AP \cos\theta$, $PL = AP \sin\theta$.

$$\therefore \quad \frac{AL}{\cos\theta} = \frac{PL}{\sin\theta} = AP,$$

i.e., $$\frac{\mathbf{x - x_1}}{\mathbf{\cos\theta}} = \frac{\mathbf{y - y_1}}{\mathbf{\sin\theta}} = \mathbf{r} \qquad \dots \qquad \dots \qquad (4)$$

where r is the distance of any point on the line from (x_1, y_1).

This, being the relation between the co-ordinates of any point on the line, is the equation of the required line.

Note 1. ***Co-ordinates of any point on the line*** **at a distance r from the point (x_1, y_1) are given by**

$$x = x_1 + r\cos\theta, \quad y = y_1 + r\sin\theta. \qquad \dots \qquad (5)$$

Sometimes $\cos\theta$, $\sin\theta$ are called ***direction-cosines*** **of the line and are denoted by l, m, so that $l^2 + m^2 = 1$.**

Substituting values proportional to $\cos\theta$, $\sin\theta$, the above equation of the line can be written as

$$\frac{x - x_1}{\lambda} = \frac{y - y_1}{\mu} = r. \qquad \dots \qquad \dots \qquad (6)$$

Now, of course, r will be proportional to the distance of any point (x, y) from (x_1, y_1) and $\lambda^2 + \mu^2 \neq 1$, and co-ordinates of any point on the line may be written as

$$x = x_1 + \lambda r, \quad y = y_1 + \mu r \qquad \dots \qquad (7)$$

and the 'm' of the line is μ/λ.

Note 2. This form of the equation of the line is sometimes called *Symmetrical form.*

Note 3. From the different forms of the equation of a line obtained above, it is clear that the equations are all of the *first degree in* x, y, that every equation involves *two independent constants* and that we can easily pass from one form to another.

3·4. Reduction of $Ax+By+C=0$ to different forms.

From the above equation, we get

$$y=-\frac{A}{B}x-\frac{C}{B} \quad (m\text{-}form).$$

Here $m=-A/B$, $c=-C/B$.

Again, $Ax+By=-C$

$$\therefore \quad \frac{x}{-C/A}+\frac{y}{-C/B}=1. \quad (Intercept\text{-}form)$$

Dividing the given equation by $\sqrt{A^2+B^2}$, we get

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=-\frac{C}{\sqrt{A^2+B^2}}.$$

(*Perpendicular-form*).

By Elementary Trigonometry we may define an angle uniquely as the least angle (positive or negative) which satisfies the equation $\tan \alpha = B/A$; so that

$$\cos \alpha=\frac{A}{\sqrt{A^2+B^2}}, \sin \alpha=\frac{B}{\sqrt{A^2+B^2}}.$$

$Ax+By+C=0$ is called the *general form* of the equation of a line.

Note. If two lines are coincident, their corresponding intercepts on the two axes will be equal. Now, writing the two equations in the intercept-forms, we see that the *conditions for the coincidence of the lines* $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are

$$-\frac{c_1}{a_1}=-\frac{c_2}{a_2} \text{ and } -\frac{c_1}{b_1}=-\frac{c_2}{b_2} \quad i.e. \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}.$$

Thus, when two lines become coincident, the coefficients of the corresponding terms of their equations are proportional. In other words, *if two equations of the first degree represent the same straight line, the ratios of corresponding coefficients must be the same.*

3·5. Line through (x_1, y_1) and (x_2, y_2).

Let the equation of any straight line be

$$y = mx + c. \quad \ldots \quad \ldots \quad (1)$$

Since it passes through (x_1, y_1), we have

$$y_1 = mx_1 + c. \quad \ldots \quad \ldots \quad (2)$$

Subtracting (2) from (1),

$$y - y_1 = m(x - x_1). \quad \ldots \quad \ldots \quad (3)$$

Since (3) passes through (x_2, y_2), we have

$$y_2 - y_1 = m(x_2 - x_1),$$

$$\therefore \quad m = \frac{y_2 - y_1}{x_2 - x_1}. \quad \ldots \quad \ldots \quad (4)$$

Substituting this value of m in (3), we get the *equation of the required line* to be

$$\mathbf{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}, \quad \ldots \quad (5)$$

$$\text{or} \quad \mathbf{\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}}. \quad \ldots \quad \ldots \quad (6)$$

Cor. From (3), it follows that the *equation of any line through* (x_1, y_1) is

$$\mathbf{y - y_1 = m(x - x_1)} \quad \ldots \quad (7)$$

m being any arbitrary constant.

Note 1. *The 'm' of the line* joining (x_1, y_1), (x_2, y_2) is

$$\frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2} = \frac{\textit{diff. of 2 ordinates}}{\textit{diff. of 2 abscissæ}}.$$

Note 2. The *equation of the line joining the origin to the point* (x_1, y_1) is

$$\frac{x}{x_1} = \frac{y}{y_1}.$$

Alternative Method :

Let the equation of any line be

$$Ax + By + C = 0. \qquad (1)$$

Since it passes through (x_1, y_1) and (x_2, y_2), we have

$$Ax_1 + By_1 + C = 0 \qquad \ldots \qquad (2)$$

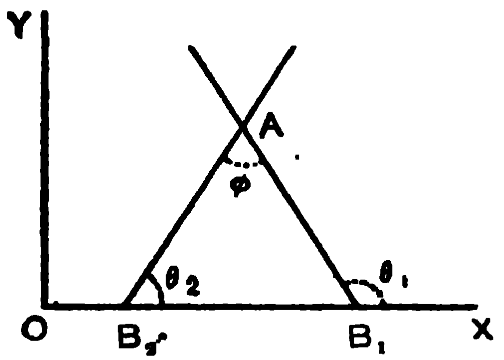
$$Ax_2 + By_2 + C = 0. \qquad \ldots \qquad (3)$$

Eliminating A, B, C between (1), (2), (3), we get the *equation of the required line* to be

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

3·6. Angle between two lines.

Let φ be the angle between the lines AB_1, AB_2, so that $\angle B_1AB_2 = \varphi$ and let $\angle AB_1X = \theta_1$ and $AB_2X = \theta_2$



$$\therefore \quad \varphi = \theta_1 - \theta_2 \qquad \ldots \qquad (1)$$

(i) Let the equations of the lines be $y = m_1x + c_1$ and $y = m_2x + c_2$ then $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.

From (1) $\tan\varphi = \tan(\theta_1 - \theta_2) = \dfrac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$.

$$\therefore \quad \mathbf{\tan\varphi = \frac{m_1 - m_2}{1 + m_1m_2}}. \qquad \ldots \qquad (1)$$

(ii) Let the equations of the lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Writing the equations as

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1} \text{ and } y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2},$$

we see that the m's of the lines are $-\frac{a_1}{b_1}$ and $-\frac{a_2}{b_2}$.

Substituting these values in formula (1), we get

$$\tan\varphi = \frac{-\frac{a_1}{b_1}+\frac{a_2}{b_2}}{1+\frac{a_1 a_2}{b_1 b_2}} = \frac{\mathbf{b_1 a_2 - a_1 b_2}}{\mathbf{a_1 a_2 + b_1 b_2}}. \quad \ldots \quad (2)$$

From above it readily follows that

$$\sin\varphi = \frac{b_1 a_2 - a_1 b_2}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}},$$

$$\cos\varphi = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}. \quad \ldots \quad (3)$$

(iii) Let the equations of the lines be

$$x\cos\alpha_1 + y\sin\alpha_1 - p_1 = 0,$$

and $$x\cos\alpha_2 + y\sin\alpha_2 - p_2 = 0.$$

Since the angle between any two lines is equal or supplementary to that between the two lines perpendicular to them,

$$\varphi = \alpha_1 - \alpha_2 \text{ or } \pi - (\alpha_1 - \alpha_2) \quad \ldots \quad (4)$$

Note. Between two straight lines there are two angles, one *acute* and the other *obtuse*. Hence in general ϕ is acute or obtuse according as right side of (1) is positive or nagative. In the fig. drawn, $\phi = \theta_1 - \theta_2$; but if the positions of AB_1, AB_2 are interchanged, $\phi = \theta_2 - \theta_1$. Taking all these things into considerations, we should write

$$\tan\phi = \pm\frac{m_1 - m_2}{1 + m_1 m_2} \text{ and } \pm\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}. \quad \ldots \quad (5)$$

3·7. Condition of Parallelism of two lines.

Two lines are parallel when the angle φ between them is zero and hence $\tan\varphi = 0$.

Hence the *condition* of parallelism of the lines

$y=m_1x+c_1$ and $y=m_2x+c_2$ is by Art. 3·6

$$\frac{m_1-m_2}{1+m_1m_2}=0 \text{ i.e. } m_1-m_2=0$$

$$\text{i.e. } \mathbf{m_1=m_2}. \quad \cdots \quad (1)$$

Similarly, the condition of parallelism of the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is $b_1a_2-b_2a_1=0$ i.e. $a_1b_2=a_2b_1$,

$$\text{i.e. } \frac{\mathbf{a_1}}{\mathbf{a_2}}=\frac{\mathbf{b_1}}{\mathbf{b_2}}. \quad \cdots \quad (2)$$

Note. From above it is clear that the *equations of two parallel lines* can be written as

$y=mx+c_1$ and $y=mx+c_2$, or $ax+by+c_1=0$ and $ax+by+c_2=0$.

Thus, we see *if two linear equations in x, y differ only in their constant terms, they represent a pair of parallel lines.*

3·8. Condition of perpendicularity of two lines.

Two lines are perpendicular when the angle φ between them is 90° and hence $\tan\varphi=\infty$.

Hence the condition of perpendicularity of the lines $y=m_1x+c_1$ and $y=m_2x+c_2$ is by Art. 3·6

$$\frac{m_1-m_2}{1+m_1m_2}=\infty \text{ i.e. } 1+m_1m_2=0$$

$$\text{i.e. } \mathbf{m_1m_2=-1}. \quad \cdots \quad (1)$$

Similarly, the condition of perpendicularity of the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is

$$\mathbf{a_1a_2+b_1b_2=0}. \quad \cdots \quad (2)$$

Note. From above it is clear that the *equations of two perpendicular lines* can be written as

$y=mx+c_1$, and $y=-\frac{1}{m}x+c_2$ or $ax+by+c_1=0$ and $bx-ay+c_2=0$.

3·9. Point of intersection of two lines.

Let the equations of the two lines be

$$a_1x + b_1y + c_1 = 0 \quad \ldots \quad (1)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots \quad (2)$$

If the lines intersect at any point, the co-ordinates of that point shall satisfy both the equations. Let (x', y') be the co-ordinates of the common point.

$$\therefore \quad a_1x' + b_1y' + c_1 = 0 \quad \ldots \quad (3).$$

$$a_2x' + b_2y' + c_2 = 0 \quad \ldots \quad (4).$$

$\therefore$ by the rule of cross-multiplication, we have

$$\frac{x'}{b_1c_2 - b_2c_1} = \frac{y'}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

$\therefore$ the co-ordinates of the common point are given by

$$x' = \frac{\mathbf{b_1c_2 - b_2c_1}}{\mathbf{a_1b_2 - a_2b_1}}, \quad y' = \frac{\mathbf{c_1a_2 - c_2a_1}}{\mathbf{a_1b_2 - a_2b_1}}.$$

Note. If $a_1b_2 - a_2b_1 = 0$, then both x', y' tend to ∞, so that the point of intersection of the lines (1) and (2) moves off to infinity and hence they become parallel. Thus $a_1b_2 - a_2b_1 = 0$, or $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ is the *condition for the parallelism* of two lines. [*Also see Art. 3·6*].

3·10. Line through the intersection of two lines.

The equation of any line through the intersection of the two lines

$$a_1x + b_1y + c_1 = 0, \quad \ldots \quad (1)$$

and

$$a_2x + b_2y + c_2 = 0 \quad \ldots \quad (2)$$

is

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0 \cdots (3)$$

where λ is any arbitrary constant.

Equation (3) being of the first degree represents a st. line. If (x', y') be the point of intersection of (1) and (2), then $a_1x' + b_1y' + c_1 = 0$ and $a_2x' + b_2y' + c_2 = 0$ and hence (x', y') satisfy (3) *i. e.* the line (3) passes through the

intersection of (1) and (2). Since the arbitrary constant λ may be so chosen that (3) may fulfil any other condition, (3) represents any st. line through the intersection of (1) and (2).

Conversely, *if the equation of any line can be put in the form*

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$$

where λ is any arbitrary constant, we conclude that *it always passes through a fixed point* viz. the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_2y + c_2 = 0$, whatever be the value of λ.

Similarly, the equation $\lambda(a_1x + b_1y + c_1) + \mu(a_2x + b_2y + c_2) = 0$ where λ and μ are any two arbitrary constants, represents any line through the intersection of the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$.

Note Let (x_1, y_1) be the point of intersection of the lines (1) and (2); then obtaining their values as in Art. 3·9 and substituting them in the equation $y - y_1 = m(x - x_1)$ [*see Cor. Art. 3·5*], where m is any arbitrary constant, *the equation of the required line* can also be obtained.

3·11. Condition for concurrence of three lines.

If the three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ pass through a common point, say (x', y'), then

$$a_1x' + b_1y' + c_1 = 0 \quad \dots \quad \dots \quad (1),$$

$$a_2x' + b_2y' + c_2 = 0 \quad \dots \quad \dots \quad (2),$$

$$a_3x' + b_3y' + c_3 = 0 \quad \dots \quad \dots \quad (3).$$

Hence eliminating x', y' from (1), (2), (3), we get the required condition for the concurrence of three lines *viz.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \quad \dots \quad \dots \quad (A).$$

Alternative Condition

If any three constants l, m, n can be found such that

$$l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)=0 \quad \text{(B)}$$

identically, then the above three lines are concurrent. From the given relation we have

$$a_3x+b_3y+c_3=-\frac{l}{n}(a_1x+b_1y+c_1)$$
$$-\frac{m}{n}(a_2x+b_2y+c_2) \qquad (4)$$

If (x', y') be the point of intersection of first two lines then $a_1x'+b_1y'+c_1=0$, and $a_2x'+b_2y'+c_2=0$ and hence from (4) we get

$$a_3x'+b_3y'+c_3=-\frac{l}{n}\times 0-\frac{m}{n}\times 0=0.$$

The third line therefore passes through (x', y'), the common point of the first two lines.

Note 1. *The above condition* (*A*) *for the concurrence can also be obtained* by substituting the co-ordinates of the point of intersection of the first two lines as given in Art. 3·8 in the equation of the third line *or thus* :

The equation of any line through the first two is

$$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0$$

or $$(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)=0.$$

This line will be identical with the third line

$$a_3x+b_3y+c_3=0$$

for some value of λ, if the third line passes through the point of intersection of the first two lines.

$\therefore$ comparing coefficients we get

$$\frac{a_1+\lambda a_2}{a_3}=\frac{b_1+\lambda b_2}{b_3}=\frac{c_1+c\lambda_2}{c_3}=-\mu, \text{ say.}$$

$\therefore$ $a_1+\lambda a_2+\mu a_3=0$, $b_1+\lambda b_2+\mu b_3=0$, $c_1+\lambda c_2+\mu c_3=0$, whence eliminating λ, μ, the required condition for the cocurrence is obtained.

Note 2. *The condition (A) can easily be deduced from condition (B) as follows :*

Since the relation (B) is identically true, the coefficients of x, y and the constant term must separately vanish. Hence we get

$$a_1 l+a_2 m+a_3 n=0,\ b_1 l+b_2 m+b_3 n=0,\ c_1 l+c_2 m+c_3 n=0\ ;$$

whence eliminating l, m, n, the condition (A) is obtained.

3·12. Condition of collinearity of three points.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be three given points. If each of them lies on the line

$$Ax+By+C=0 \quad \ldots \quad (1),$$

then

$$Ax_1+By_1+C=0 \quad \ldots \quad (2),$$

$$Ax_2+By_2+C=0 \quad \ldots \quad (3),$$

$$Ax_3+By_3+C=0 \quad \ldots \quad (4).$$

Eliminating A, B, C from (2), (3), (4), we get the required condition to be

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0. \quad \ldots \quad (5)$$

Note. *The above condition is both necessary and sufficient.* Assuming that relation (5) holds, we get on simplifying this determinantal relation and re-arranging the terms,

$$(x_1-x_2)(y_1-y_3)-(x_1-x_3)(y_1-y_2)=0,$$

$$\text{or,}\quad \frac{y_1-y_2}{x_1-x_2}=\frac{y_1-y_3}{x_1-x_3},$$

which shows that the 'm's of the two lines joining (x_1, y_1) to (x_2, y_2) and to (x_3, y_3) are the same both in magnitude and direction, which can happen only when the 3 points are collinear. Thus, the sufficiency of the condition is proved.

3·13. Illustrative Examples.

Ex. 1. *Find the equation of the line through (x_1, y_1)*

(i) *parallel to the line $ax+by+c=0$,*

(ii) *perpendicular to the line $ax+by+c=0$.*

(i) Equation of any line parallel to the given line is $ax+by+k=0$. Since it passes through (x_1, y_1),

$$\therefore \quad ax_1+by_1+k=0 \quad i.e. \quad k=-(ax_1+by_1).$$

Hence the reqd. equation of the line is

$$ax+by-(ax_1+by_1)=0.$$

$$i.e. \quad a(x-x_1)+b(y-y_1)=0.$$

Otherwise: Let the equation of the line through (x_1, y_1) be $y-y_1=m(x-x_1)$; since it is parallel to the given line whose 'm' is $-a/b$, $\therefore m=-a/b$

$\therefore$ the reqd. line is $y-y_1=-\frac{a}{b}(x-x_1)$,

$i.e.$ $a(x-x_1)+b(y-y_1)=0.$

(ii) Any line perpendicular to the given line is $bx-ay+k=0$. Since it passes through (x_1, y_1) $\therefore$ $bx_1-ay_1+k=0$ $i.e.$ $k=ay_1-bx_1$

$\therefore$ the reqd. line is $bx-ay+ay_1-bx_1=0$,

$$i.e. \quad \frac{x-x_1}{a}=\frac{y-y_1}{b}.$$

The above equation can also be obtained by assuming the equation through (x_1, y_1) in the 'm' form as in (i) viz. $y-y_1=m(x-x_1)$ and then introducing the condition that this line and the given line are perp. $i.e.$ $m\times(-a/b)=-1$ $i.e.$ $m=b/a$. Now substitute this value of m.

Ex. 2. *Show that the equations of the lines which pass through (x_1, y_1) and are inclined at an angle ϕ with the line $ax+by+c=0$ is*

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ a\sin\phi\pm b\cos\phi & b\sin\phi\pm a\cos\phi & 0 \end{vmatrix}=0,$$

Let the equation of the line be

$$Ax+By+C=0. \quad \dots \quad \dots \quad (1)$$

Since it passes through (x_1, y_1)

$$\therefore \quad Ax_1+By_1+C=0. \quad \dots \quad \dots \quad (2)$$

Since it makes an angle ϕ with $ax+by+c=0$.

$$\therefore \quad \frac{\sin\phi}{\cos\phi}=\tan\phi=\pm\frac{Ab-Ba}{Aa+Bb}.$$

$$\therefore \quad A(a\sin\phi\mp b\cos\phi)+B(b\sin\phi\pm a\cos\phi)=0 \quad \dots \quad (3)$$

Eliminating A, B, C from (1), (2) and (3), the reqd. result is obtained.

Ex. 3. *Show that the perps. dropped from the vertices of a triangle on the opposite sides are concurrent.*

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the triangle.

The 'm' of BC is $\dfrac{y_2 - y_3}{x_2 - x_3}$

$\therefore$ equation of the line through A perp. to BC is

$$y - y_1 = m(x - x_1) \cdots (1) \text{ where}$$

$$m.\frac{y_2 - y_3}{x_2 - x_3} = -1 \quad i.e. \quad m = -\frac{x_2 - x_3}{y_2 - y_3}.$$

Substituting this value of m in (i). and simplifying, the equation of the perp. from A to BC, is

$$x(x_2 - x_3) + y(y_2 - y_3) - x_1(x_2 - x_3) - y_1(y_2 - y_3) = 0 \quad \ldots (2)$$

Similarly the perps. from B and C on opposite sides are

$$x(x_3 - x_1) + y(y_3 - y_1) - x_2(x_3 - x_1) - y_2(y_3 - y_1) = 0 \cdots (3)$$

$$x(x_1 - x_2) + y(y_1 - y_2) - x_3(x_1 - x_2) - y_3(y_1 + y_2) = 0 \ldots (4)$$

On adding (2), (3), (4), we find that their sum vanishes identically. $\therefore$ by Art. 3·11 (here $l = m = n = 1$), we conclude the three lines meet in a point.

Note. The point of concurrence is called the *orthocentre* of the triangle.

Ex. 4. *Find the diagonals of the parallelogram whose sides are*

$$ax + by + c = 0 \ (1), \qquad a'x + b'y + c = 0 \quad \ldots \quad (2),$$

$$ax + by + c' = 0 \ (3), \qquad a'x + b'y + c' = 0 \quad \ldots \quad (4),$$

and hence obtain the condition that they should be at right angles.

Any line through (1) and (2) is

$$ax + by + c + \lambda(a'x + b'y + c) = 0.$$

If it be a diagonal, it must pass through (α, β) the point of intersection of (3), (4).

$$\therefore \quad a\alpha + b\beta + c + \lambda(a'\alpha + b'\beta + c) = 0.$$

$$\therefore \quad -c' + c + \lambda(-c' + c) = 0. \quad \ldots \quad (5)$$

since (α, β) lying on (3) and (4)

$$a\alpha + b\beta + c' = 0, \quad a'\alpha + b'\beta + c' = 0.$$

$\therefore$ from (5), $\lambda = -1$.

$\therefore$ the corresponding diagonal will be

$$(a-a')x+(b-b')y=0. \quad \dots \quad (6)$$

Similarly, the other diagonal will be

$$(a+a')x+(b+b')y+c+c'=0. \quad \dots \quad (7)$$

Condition that (6) and (7) should be perp. is by Art. 3·8

$$(a-a')(a+a')+(b-b')(b+b')=0,$$

i.e. $$a^2+b^2=a'^2+b'^2.$$

Examples IIIA

1. Find the equation of the line joining the points

(i) $(a \cos \alpha, a \sin \alpha)$, $(a \cos \beta, a \sin \beta)$,

(ii) $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$,

(iii) $(a \cos \varphi_1, b \sin \varphi_1)$, $(a \cos \varphi_2, b \sin \varphi_2)$.

2. Show that the equation

$$(x-x_1)(y-y_2) = (x-x_2)(y-y_1)$$

represents the line joining (x_1, y_1), (x_2, y_2).

3. Show that the equation of the line, the portion of which intercepted between the axes is bisected at (x_1, y_1) is

$$\frac{x}{2x_1}+\frac{y}{2y_1}=1.$$

4. Calculate the angle included between the lines

(i) $ax-by+c=0$, $(a-b)x-(a+b)y+d=0$,

(ii) $x \cos 55^\circ - x + y \sin 55^\circ + \sqrt{3}+1=0$,

$x \sin 35^\circ - x + y \cos 35^\circ + \sqrt{3}-1=0$.

5. If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of a triangle, show that the angle A is acute or obtuse according as

$$(x_1-x_2)(x_1-x_3)+(y_1-y_2)(y_1-y_3) \text{ is}$$

positive or negative.

6. Show that the three lines

$$x \cos\left(a + \frac{r}{3}\pi\right) + y \sin\left(a + \frac{r}{3}\pi\right) = a, \quad (r = 2, 4, 6)$$

form an equilateral triangle.

7. Find the equations of the line through (x_1, y_1) respectively perpendicular to

(i) $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1,$

(ii) $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

8. Find the co-ordinates of the points of intersection of the following pair of lines

(i) $y = m_1 x + \dfrac{a}{m_1}, \; y = m_2 x + \dfrac{a}{m_2},$

(ii) $\dfrac{x}{a}\cos\varphi + \dfrac{y}{b}\sin\varphi = 1, \; \dfrac{x}{a}\cos\varphi' + \dfrac{y}{b}\sin\varphi' = 1.$

9. Show that the lines $(a+b)x + (a-b)y - 2ab = 0,$ $(a-b)x + (a+b)y - 2ab = 0,$ and $x + y = 0$ form an isosceles triangle whose vertical angle is $2\tan^{-1} a/b$.

10. Show that the equation of the line through (x', y') and

(i) parallel to $ax + by + c = 0$

is $$\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ b & -a & 0 \end{vmatrix} = 0$$

(ii) perpendicular to $ax + by + c = 0$

is $$\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ a & b & 0 \end{vmatrix} = 0$$

11. Show that the perpendicular from the origin upon the join of $(a\cos\alpha, a\sin\alpha)$, $(a\cos\beta, a\sin\beta)$ bisects it.

12. (i) Find the equation of the line which passes through the point of intersection of the lines $3x+2y-7=0$ and $x-4y+6=0$ and through the point (1, 1).

(ii) Show that the equation of the line through (α, β) and through the point of intersection of the lines

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0$$

is
$$\frac{a_1x+b_1y+c_1}{a_1\alpha+b_1\beta+c_1}=\frac{a_2x+b_2y+c_2}{a_2\alpha+b_2\beta+c_2}.$$

[*C. U. 1943*]

13. Find the equation of the line joining the point of intersection of the lines

$$x+3y+2=0,\ 2x-y-3=0$$

to the point of intersection of

$$7x-y-3=0,\ 2x-5y-15=0.$$

14. Find the equation of the line which passes through the point of intersection of

(i) the lines $2x-3y+4=0$, $3x+4y-5=0$ and is perp. to the line $6x-7y+8=0$; [*C. U. 1944*]

(ii) the lines $x-y+1=0$, $3x+y-5=0$ and is parallel to $7x-8y+13=0$;

(iii) the lines $7x+13y-87=0$, $5x-8y+7=0$ and makes equal intercepts on the co-ordinate axes.

15. Show that the equation to the line that passes through the point of intersection of the lines

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0$$

and is (i) parallel to the line $a_3x+b_3y+c_3=0$

is
$$\frac{a_1x+b_1y+c_1}{a_1b_3-a_3b_1}=\frac{a_2x+b_2y+c_2}{a_2b_3-a_3b_2};$$

(ii) perp. to the line $a_3x+b_3y+c_3=0$

is
$$\frac{a_1x+b_1y+c_1}{a_1a_3+b_1b_3}=\frac{a_2x+b_2y+c_2}{a_2a_3+b_2b_3}.$$

16. (i) Prove that the three lines $2x-7y+10=0$, $3x-2y-1=0$ and $x-12y+21=0$ meet in a point.
[*C. U. 1943*]

(ii) Find for what value of k will the three lines $x-y+5=0$, $x+y-1=0$, $kx-y+13=0$ be concurrent.

Find also the point of concurrence.

17. If $a+b+c=0$, show that the three lines

$$ax+by+c=0, \quad bx+cy+a=0, \quad cx+ay+b=0$$

are concurrent.

18. Show that the four lines

$$ax+by+c=0,\ (a+a')x+(b+b')y+(c+c')=0,$$
$$a'x+b'y+c'=0,\ (a-a')x+(b-b')y+(c-c')=0$$

meet in a point

19. If the three lines

$$ax+a^2y+1=0,\ bx+b^2y+1=0,\ cx+c^2y+1=0$$

are concurrent, show that at least two of the three constants a, b, c are equal.

20. In any triangle, show that

(i) the medians are concurrent ;

(ii) the lines drawn through the mid-points of the sides and perp. to them are concurrent.

21. Show that the three points

$$(a_1, b_1), (a_2, b_2), (a_3, b_3)$$

will be collinear if and only if the three lines

$$a_1x+b_1y-1=0,\ a_2x+b_2y-1=0,\ a_3x+b_3y-1=0$$

are concurrent.

22. When the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear, show that the line of collinearity is given by any of the three *equivalent* equations

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0, \quad \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0, \quad \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}=0.$$

23. Prove that the points $(a, bc-a^2)$, $(b, ca-b^2)$, $(c, ab-c^2)$ are collinear and find the intercept on the axis of y by the line on which they lie.

24. If the points (a, b), (a', b'), $(a-a', b-b')$ are collinear show that their join passes through the origin and $ab' = a'b$. [*C.U. 1938*]

25. If the coefficients of x, y in the equation of a line vary, subject to the condition that their sum is constant, show that the line passes through a fixed point.

26. If the variable co-efficients λ, μ, ν in the equation of the line $\lambda x + \mu y + \nu = 0$ be connected by a relation of the form $\lambda a + \mu b + \nu c = 0$, where a, b, c are constants, then the variable line always passes through a fixed point.

27. Show that the two sets of points

(i) $(0, \frac{11}{3})$, $(1, \frac{19}{3})$, $(-1, 1)$

and (ii) $(0, \frac{5}{8})$, $(2, -\frac{1}{8})$, $(\frac{2}{3}, \frac{3}{8})$

lie on two lines which are mutually perpendicular.

28. Find the co-ordinates of the middle points of three diagonals of the complete quadrilateral formed by the lines $x = 0$, $y = 0$, $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{a'} + \dfrac{y}{b'} = 1$ and show that these points are collinear. [*C.U. 1941*]

29. Find the equations of the diagonals of the parallelogram formed by the lines

$$4x - 5y - 7 = 0,\ 4x - 5y - 14 = 0,$$
$$3x + 7y - 8 = 0,\ 3x + 7y - 12 = 0.$$

30. Show that the diagonals of the parallelogram formed by the lines

$$\frac{x}{a} + \frac{y}{b} = 1,\quad \frac{x}{b} + \frac{y}{a} = 1,\quad \frac{x}{a} + \frac{y}{b} = 2,\quad \frac{x}{b} + \frac{y}{a} = 2$$

are at right angles. [*C. U. 1939*]

31. Find the equations of two lines through the point $(3, -1)$ which make 45° with $2x - y - 2 = 0$.

32. (*a*) Find the orthocentre of the triangle whose sides are

(i) $2x+3y-6=0,\ 6x-y-18=0,\ x-y-2=0$;

(ii) $y=m_r x+\dfrac{a}{m_r}, \qquad (r=1, 2, 3)$;

(*b*) Find the orthocentre of the triangles whose vertices are $(3, 0), (0, 2), (4, 6)$.

33. Find the area of the triangle whose sides are

(i) $x+2y-5=0,\ x-y+1=0,\ 2x+y-7=0$;

(ii) $y=m_1x+c_1,\ \ y=m_2x+c_2,\ \ x=0$;

(iii) $y=m_1x+c_1,\ \ y=m_2x+c_2,\ \ y=m_3x+c_3$.

34. Show that the line joining the points of intersection of two pairs of lines

$$\left.\begin{array}{l} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{array}\right\} \qquad \left.\begin{array}{l} a_3x+b_3y+c_3=0 \\ a_4x+b_4y+c_4=0 \end{array}\right\}$$

is
$$\frac{a_1x+b_1y+c_1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}}=\frac{a_2x+b_2y+c_2}{\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}}.$$

[*To determine* λ *in the equation of any line through the first pair, use the condition that this line and the second pair are concurrent.*]

35. If $A(\alpha_1, \beta_1), B(\alpha_2, \beta_2), C(\alpha_3, \beta_3), D(\alpha_4, \beta_4)$ be four given points, prove that the locus of a variable point P which moves so as to satisfy the condition

$$\triangle ABP \pm \triangle CDP = \text{const.}$$

consists of two right lines.

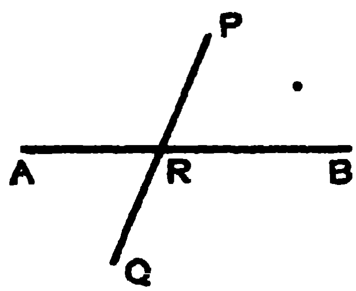
3·14. Points on the same or opposite sides of a given line.

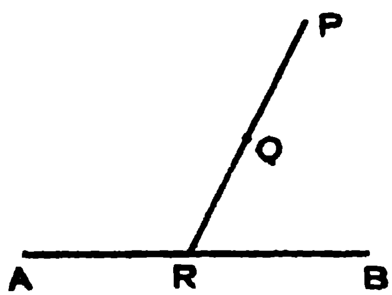
Let the equation of the given line AB be $ax+by+c=0$ and let the co-ordinates of the given points P, Q be (x_1, y_1) and (x_2, y_2).

Suppose PQ cuts AB in R, such that $\dfrac{PR}{RQ}=\dfrac{m}{n}$.

$\therefore$ when P, Q are on the *opposite sides* of AB,

co-ordinates of R are $\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}$ [*Art.* 2·5]





Since R lies on AB.

$$\therefore \quad a\left(\frac{mx_2+nx_1}{m+n}\right)+b\left(\frac{my_2+ny_1}{m+n}\right)+c=0$$

from which we get

$$\frac{ax_1+by_1+c}{ax_2+by_2+c}=-\frac{m}{n}$$

$=$ a negative quantity.

$\therefore$ ax_1+by_1+c and ax_2+by_2+c have opposite signs.

Similarly, when P, Q are on the *same side* of AB

co-ordinates of R are $\dfrac{mx_2-nx_1}{m-n}, \dfrac{my_2-ny_1}{m-n}$. [*Art.* 2.5]

Now, proceeding as before, it can be shown in this case that

$$\frac{ax_1+by_1+c}{ax_2+by_2+c}=\frac{m}{n}$$

$=a$ positive quantity.

$\therefore$ ax_1+by_1+c and ax_2+by_2+c have same signs.

Thus, *the two points* (x_1, y_1) *and* (x_2, y_2) *lie on the same or opposite sides of the line* $ax+by+c=0$ *according as*

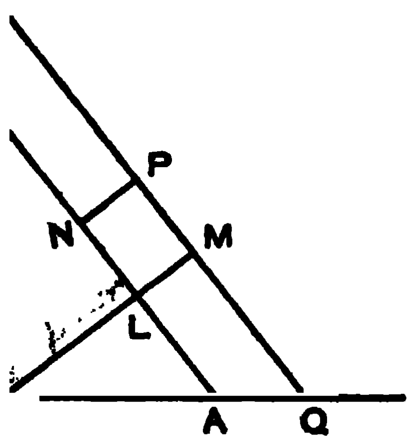
the quantities $ax_1 + by_1 + c$ *and* $ax_2 + by_2 + c$ *have same or opposite signs.*

That *side* of the line $ax + by + c = 0$ is usually called *positive*, the co-ordinates of any point of which when substituted in $ax + by + c$ make it positive, the other *side*, being called *negative*. If the constant 'c' in the equation $ax + by + c = 0$ is positive the origin $(0, 0)$ is situated on the positive side of the line and if negative, on the nagative side.

Cor. *The point* (x_1, y_1) *and the origin are on the same or opposite sides of the line* $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and c have the *same* or *opposite signs.*

3·15. Length of the perpendicular.

To find the length of the perpendicular dropped from a given point upon a given line.



Let P be the given point (x', y') and AB be the given line. Draw PN perp. on AB.

(i) Let the equation of the line be $x \cos a + y \sin a - p = 0$ ··· (1)

Let OL be the perp. from O on this line, so that $OL = p$ and $\angle XOM = a$.

Through P draw a line PQ parallel to the given line meeting OL produced in M and let $OM = p'$.

Then the equation of PQ is by Art. 3·3 (C)

$$x \cos a + y \sin a - p' = 0.$$

Since it passes through (x', y')

$$\therefore \quad x' \cos a + y' \sin a - p' = 0.$$

$$\therefore \quad p' = x' \cos a + y' \sin a.$$

$\therefore$ the reqd. length of the perpendicular

$$= PN = ML = OM - OL = p' - p$$
$$= \mathbf{x' \cos \alpha + y' \sin \alpha - p} \quad \ldots \quad (1)$$

(ii) Let the equation of the line be given in the form

$$Ax + By + C = 0.$$

Let us first express the equation in the form $x \cos \alpha + y \sin \alpha - p = 0$ as in Art. 3·4. Dividing the given equation by $\sqrt{A^2 + B^2}$, we have

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y + \frac{C}{\sqrt{A^2+B^2}} = 0,$$

so that $\cos \alpha = \dfrac{A}{\sqrt{A^2+B^2}}$, $\sin \alpha = \dfrac{B}{\sqrt{A^2+B^2}}$,

$$p = \frac{-C}{\sqrt{A^2+B^2}}.$$

$\therefore$ the length of the perp. from (x', y') on it

$$= \frac{\mathbf{Ax' + By' + C}}{\sqrt{\mathbf{A^2 + B^2}}}. \quad \ldots \quad (2)$$

Thus *the length of the perpendicular from (x', y') upon the line $Ax + By + C = 0$ is obtained by substituting x' and y' for x and y in the left-side of the equation and dividing the result of substitution by the square root of the sum of the squares of the coefficients of x and y.*

Note 1. *Sign of the perpendicular.*

From the fig, it is clear that if P is on the other side of the line AB, length of the perp. will be $p - p'$ *i.e.* $-(x' \cos \alpha + y' \sin \alpha - p)$. So in the general expressions for the lengths of the perp. in (1) and (2) we should attach $\pm$ signs before them. The following convention is usually adoped in distinguishing the signs :

If the equation be so written that the absolute term is positive, so that the origin is on the positive side of the line, the perp. from any point will be positive or negative according as the point is on the same side of line or on the opposite side.

Note 2. The *length of the perpendicular from the origin* upon

$$Ax+By+C=0 \quad \text{is} \quad \frac{C}{\sqrt{A^2+B^2}}.$$

Note 3. Alternative methods of finding the length of the perpendicular are indicated in Ex. 1, Art. 3·20.

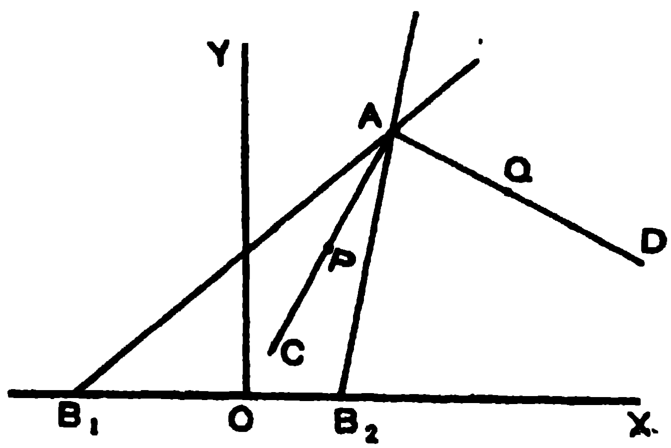
3·16. Bisectors of angles between two lines.

Let the equations of the two lines AB_1, AB_2 be

$$a_1x+b_1y+c_1=0$$

and $$a_2x+b_2y+c_2=0$$

the equations being so written that c_1 and c_2 are both positive.



Since the bisectors are the locus of points such that the magnitudes of the perps. from them on the two lines are equal, if (x', y') be the co-ordinates of any point P on the *internal* bisector AC, then

$$\frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x'+b_2y'+c_2}{\sqrt{a_2^2+b_2^2}}. \qquad (1)$$

Since in this case (x', y') and the origin are on the same sides of both the lines AB_1, AB_2, the signs of both sides of (1) are the same. [*See Notes 1, 2 of Art. 15*],

Similarly, if (x', y') be the co-ordinates of any point Q on the *external* bisector AD, we have

$$\frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x'+b_2y'+c_2}{\sqrt{a_2^2+b_2^2}} \qquad (2)$$

In this case, since (x', y') and the origin are on opposite sides of one of the two lines *viz.* AB_2, the two sides of (2) are of opposite signs. [*See Notes 1, 2 of Art. 15.*]

The relations (1) and (2) show that equations of the two bisectors are given by

$$\frac{\mathbf{a_1x+b_1y+c_1}}{\sqrt{\mathbf{a_1}^2+\mathbf{b_1}^2}}=\pm\frac{\mathbf{a_2x+b_2y+c_2}}{\sqrt{\mathbf{a_2}^2+\mathbf{b_2}^2}}$$

(the upper sign being the bisector of the angle in which the origin lies).

Similarly, when the equations of the lines are

$$x\cos\alpha_1+y\sin\alpha_1-p_1=0,$$
$$x\cos\alpha_2+y\sin\alpha_2-p_2=0$$

the equations of the bisectors are

$$\mathbf{x\cos\alpha_1+y\sin\alpha_1-p_1=\pm(x\cos\alpha_2+y\sin\alpha_2-p_2).}$$

Note. It can be easily verified that these two bisectors are perpendicular to each other.

3·17. Parametric Representation.

When the co-ordinates of any point on a locus are expressed in terms of a single variable called a *parameter*, they are called *parametric co-ordinates*. The two equations so obtained represent the locus and sometimes they are called *parametric equations* of the curve. The elimination of the parameter leads to the Cartesian equation.

From the Note 1 of Art. 3·3(D), it is clear that any point on a straight line can be represented by

$$\mathbf{x=a+bt,\quad y=c+dt} \quad \ldots \quad (1)$$

where t is a variable parameter.

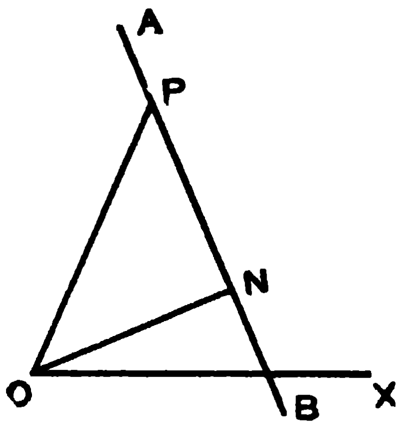
From (1), we have $\dfrac{x-a}{b}=\dfrac{y-c}{d}$.

This shows that the *line* (1) *passes through* (a, b) *and its* 'm' *is* d/b.

Elimination of t between the two relations of (1) obviously leads to a first degree equation which represents a straight line.

3·18. Polar equation of a line.

Let p be the length of the perp. ON from O on the given line AB and let $\angle XON = \alpha$.



Let (r, θ) be the polar co-ordinates of any point P on the line AB.

From $\triangle OPN$, $p = ON$

$= OP \cos PON = r \cos (\theta - \alpha)$.

$\therefore$ the polar equation of the line AB is

$$\mathbf{r \cos (\theta - \alpha) = p} \qquad \dots \quad (1),$$

where (p, α) are the polar co-ordinates of the foot of the perp. from the origin on the line.

Note 1. The equation (1) can be immediately obtained by transforming the Cartesian equation $x \cos \alpha + y \sin \alpha - p = 0$ into polar form by writing $r \cos \theta$ for x and $r \sin \theta$ for y.

Note 2. The equation (1) is equivalent to

$$r \cos \theta \cos \alpha + r \sin \theta \sin \alpha = p.$$

Dividing by p and writing A for $\cos \alpha/p$ and B for $\sin \alpha/p$, the general equation of a line in polar co-ordinates can be written as

$$\frac{1}{r} = A \cos \theta + B \sin \theta. \qquad \dots \qquad \dots \quad (2)$$

3·19. Polar equation of a line joining two points.

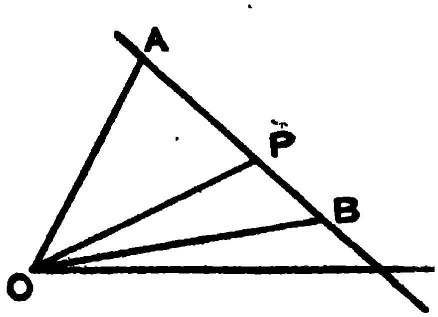
Let (r_1, θ_1), (r_2, θ_2) be the co-ordinates of the given points A, B and (r, θ) those of any point P on AB.

Then $\triangle AOB = \triangle AOP + \triangle POB$,

i.e. $\frac{1}{2} r_1 r_2 \sin (\theta_1 - \theta_2)$

$= \frac{1}{2} r_1 r \sin (\theta_1 - \theta)$

$+ \frac{1}{2} r_2 r \sin (\theta - \theta_2)$.



Dividing by $\frac{1}{2} r r_1 r_2$, we have the reqd. equation of the line is

$$\frac{\sin (\theta_1 - \theta_2)}{r} = \frac{\sin (\theta_1 - \theta)}{r_2} + \frac{\sin (\theta - \theta_2)}{r_1} \quad \dots \quad (1)$$

Alternative Method.

The polar equation of a line can be written as

$$r^{-1} - B \sin \theta - A \cos \theta = 0.$$

If it pass through (r_1, θ_1) and (r_2, θ_2), we have

$$r_1^{-1} - B \sin \theta_1 - A \cos \theta_1 = 0,$$
$$r_2^{-1} - B \sin \theta_2 - A \cos \theta_2 = 0.$$

Eliminating $-B$, $-A$, we have the reqd. equation

$$\begin{vmatrix} r^{-1} & \sin \theta & \cos \theta \\ r_1^{-1} & \sin \theta_1 & \cos \theta_1 \\ r_2^{-1} & \sin \theta_2 & \cos \theta_2 \end{vmatrix} = 0. \quad \ldots \quad (2)$$

Forms (1) and (2) can easily be shown to be identical.

3·20. Illustrative Examples.

Ex. 1. *Find the length of the perp. from (x', y') on the line $ax+by+c=0$ and the co-ordinates of the foot of the perp.*

Let $N(\alpha, \beta)$ be the foot of the perp. and let P be (x', y'). Then $a\alpha+b\beta+c=0$, and equation of PN, being the join of the two points is

$$\frac{x-x'}{x'-\alpha} = \frac{y-y'}{y'-\beta}. \quad \ldots \quad \ldots \quad (1)$$

Since PN passes through (x', y') and is perp. to $ax+by+c=0$, its equation is

$$\frac{x-x'}{a} = \frac{y-y'}{b}. \quad [\textit{Art. 3·13 Ex. 1}] \ldots \quad (2)$$

From (1) and (2), $\dfrac{x'-\alpha}{a} = \dfrac{y'-\beta}{b} = \dfrac{ax'+by'-(a\alpha+b\beta)}{a^2+b^2}$

$$= \frac{ax'+by'+c}{a^2+b^2} = u \text{ say.}$$

$$\therefore \quad x'-\alpha = au, \quad y'-\beta = bu. \quad \ldots \quad (3)$$

$$\therefore \quad PN^2 = (x'-\alpha)^2 + (y'-\beta)^2 = u^2(a^2+b^2).$$

$$\therefore \quad PN = \frac{ax'+by'+c}{\sqrt{a^2+b^2}}.$$

From (3), $\alpha = x' - au$; $\beta = y' - bu$.

Ex. 2. *Find the area of the triangle whose vertices are $A(x_1, y_1)$ $B(x_2, y_2)$, $C(x_3, y_3)$, using the relation area $= \frac{1}{2}$. base $\times$ altitude.*

Let AD be the length of perp. from A on BC.

Then $\quad \Delta = \frac{1}{2} BC.AD.$

Now, $\quad BC = \sqrt{\{(x_2 - x_3)^2 + (y_2 - y_3)^2\}}. \quad \ldots \quad (1)$

Equation of BC is $\dfrac{x - x_2}{x_2 - x_3} = \dfrac{y - y_2}{y_2 - y_3}$.

i.e. $\quad x(y_2 - y_3) - y(x_2 - x_3) + x_2 y_3 - x_3 y_2 = 0.$

$$\therefore \quad AD = \frac{x_1(y_2 - y_3) - y_1(x_2 - x_3) + x_2 y_3 - x_3 y_2}{\sqrt{\{(x_2 - x_3)^2 + (y_2 - y_3)^2\}}}. \quad (2)$$

From (1) and (2), the reqd. result follows.

Examples III(B)

1. Show that the origin and the point (1, 2) are on opposite sides of the line $2x + 3y - 4 = 0$. Find the point where the join of (0, 0) to (1, 2) meets the line.

2. Prove that the origin is inside the triangle whose vertices are $\quad (2, 1), (3, -2), (-4, -1)$.

3. If p_1, p_2 be the perpendiculars from the origin upon the lines $x \sin\theta + y \cos\theta = \frac{1}{2}a \sin 2\theta$ and $x \cos\theta - y \sin\theta = a \cos 2\theta$, then $4p_1^2 + p_2^2 = a^2$. [C.U.]

4. If d_1, d_2, d_3 be the distances of a certain point from the three lines $x \cos a_r + y \sin a_r = p_r$, $(r = 1, 2, 3)$

then $\quad \Sigma(p_1 + d_1) \sin (a_2 - a_3) = 0.$

5. Find the distance between the parallel lines

$$3x + 4y - 7 = 0, \ 3x + 4y - 12 = 0.$$

6. Show that the perpendiculars dropped from any point on the line $9x - 13y = 6$ upon the two lines $3x + 4y = 1$ and $24x + 7y = 11$ are equal.

7. Determine a point equally distant from the three lines

$$x \cos 30^\circ + y \sin 30^\circ = 2, \quad x \cos 45^\circ + y \sin 45^\circ = 2,$$
$$x \cos 60^\circ + y \sin 60^\circ = 2.$$

8. Prove that the four right lines

$$x \cos a + y \sin a = p, \qquad -x \sin a + y \cos a = p$$
$$x \cos a + y \sin a + p = 0, \qquad x \sin a - y \cos a = p$$

make up a square.

[*The perps. from the origin on the four lines are of the same length.*]

9. Show that the quadrilateral formed by the four lines

$$y = mx + a\sqrt{1+m^2}, \quad y = m'x + a\sqrt{1+m'^2},$$
$$y = mx - a\sqrt{1+m^2}, \quad y = m'x - a\sqrt{1+m'^2}$$

is a square or rhombus, according as $mm' = -1$ or $\neq -1$.

10. Find the equation of the line which lies mid-way between the lines $ax + by + c = 0$, $ax + by + c' = 0$.

11. Find the distance from point $(1, 2)$ to the line $3x + y + 4 = 0$ measured parallel to the line $3x - 4y + 8 + 0$.

12. Find the co-ordinates of the foot of the perp. let fall from $(2, 3)$ upon the line $4x + 3y - 7 = 0$ and hence deduce the length of the perp. from the point on the line.

13. Prove that the co-ordinates of the foot of the perp. from (x_1, y_1) on the line $ax + by + c = 0$ are

$$\frac{b(bx_1 - ay_1) - ac}{a^2 + b^2}, \quad \frac{a(ay_1 - bx_1) - bc}{a^2 + b^2}.$$

14. Show that the feet of the perps. from the origin to the lines

$$x + y - 4 = 0, \quad x + 5y - 26 = 0, \quad 15x - 27y - 424 = 0$$

all lie on a line and find the equation of this line.

15. Find the equations of the bisectors of the angles between the lines $3x + 4y + 4 = 0$ and $5x + 12y + 4 = 0$.

16. Find the equations of the lines which pass through the origin and are equally inclined to the lines

$$2x + 3y - 5 = 0, \quad 3x + 2y - 7 = 0.$$

17. Prove analytically that the internal and external bisectors of an angle are at right angles.

18. Find the internal bisectors of the angles of the triangle whose sides are $x=0$, $y=0$ and $3x+4y-12=0$ and show that they meet in a point.

19. In any triangle, prove that

(i) the three internal bisectors are concurrent ;

(ii) two external bisectors and one internal bisector are concurrent.

20. Find the in-centre of the triangle

(i) whose sides are

$$3x+4y-12=0,\ 12x-5y=0,\ 4x+3y-10=0\ ;$$

(ii) whose vertices are (0, 0), (0, 3), (4, 0).

21. If the distances of a point from two fixed lines $x\cos\alpha+y\sin\alpha=p$ and $x\cos\beta+y\sin\beta=q$ are in a constant ratio, show that the locus of the point is a line passing through the intersection of the given lines.

22. Find the locus of the foot of the perp. from the point $(a, 0)$ to the line $x-ty+at^2=0$, where t varies.

23. If p, q, r the lengths of the perps. from a point on the sides of a triangle be connected by a relation $\lambda p+\mu q+\nu r=0$, where λ, μ, ν are constants, then the locus of the point is a line.

24. The distance of the point (α, β) from each of the two lines through the origin is d. Prove that the equation of the lines is $$(\beta x-\alpha y)^2=d^2(x^2+y^2).$$

25. The parametric co-ordinates of a point P are

$$x=a\,(\cos t+\sin t)^2,\quad y=a\,(\cos t-\sin t)^2,$$

where t is a variable parameter. Show that the locus of P is a line.

26. Prove that $x=\dfrac{a+bt}{p+qt}$, $y=\dfrac{c+dt}{p+qt}$ represents a line.

27. Show that the lines

$$x = a_1 + b_1 s,\ y = c_1 + d_1 s,$$
$$x = a_2 + b_2 t,\ y = c_2 + d_2 t$$

are (i) parallel if $\quad b_1 d_2 - b_2 d_1 = 0$;

(ii) perpendicular if $b_1 b_2 + d_1 d_2 = 0$.

28. (i) Show that the two lines

$$\frac{k}{r} = A \cos \theta + B \sin \theta, \quad \frac{k'}{r} = A \cos \theta + B \sin \theta$$ are parallel.

(ii) Verify that the two lines

$$r \cos (\theta - \alpha) = p \quad \text{and} \quad r \sin (\theta - \alpha) = p'$$

are perpendicular.

29. Find the point of intersection of the lines

$$\frac{k}{r} = \cos \theta - \cos (\theta - \alpha), \quad \frac{k}{r} = \cos \theta - \cos (\theta - \beta).$$

30. Show that the three lines

$$r \cos (\theta - \alpha_1) = p_1,\ r \cos (\theta - \alpha_2) = p_2,\ r \cos (\theta - \alpha_3) = p_3$$

will be concurrent, if and only if

$$\begin{vmatrix} \cos \alpha_1 & \sin \alpha_1 & p_1 \\ \cos \alpha_2 & \sin \alpha_2 & p_2 \\ \cos \alpha_3 & \sin \alpha_3 & p_3 \end{vmatrix} = 0.$$

31. A line moves so that the sum of the reciprocals of its intercepts on two axes is constant. Show that it passes through a fixed point.

32. A line moves so that the algebraic sum of the perp. distances upon it from a number of fixed points is always zero. Show that it must always pass through a fixed point.

[*Take the equation of the line as* $x \cos \alpha + y \sin \alpha - p = 0$]

33. Show that the locus of a point which moves in such a way that the difference of its distances from two fixed

mutually perp. lines is equal to its distance from a fixed line, is a straight line.

34. Find the locus of the foot of the perp. from the origin upon a line which always passes through a given point (a, b).

35. A line passes through a fixed point (x_1, y_1). Show that the equation of the locus of the middle point of it intercepted between the axes is

$$\frac{x_1}{2x} + \frac{y_1}{2y} = 1.$$

CHAPTER IV

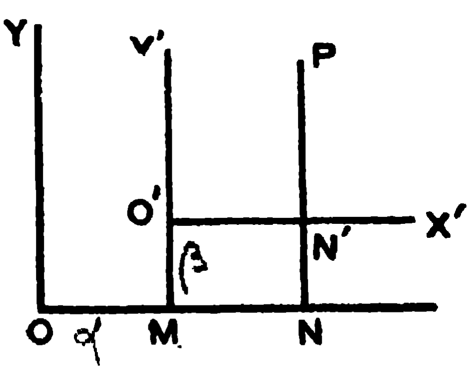
TRANSFORMATION OF CO-ORDINATES

4·1. The co-ordinates of a point depend upon the origin and the axes of co-ordinates chosen. A point being given in position, its co-ordinates with reference to one set of axes will change as soon as a new set of axes are chosen. It is also evident that the equation of a curve will also change by such transformation. In the discussion of problems it is very aften found advantageous to change the origin or the directions of axes or both. Either of these processes is called a *transformation of co-ordinates.* The formulæ for such transformation will be established in the following articles.

4·2. Change of origin.

To pass from one set of rectangular axes to a parallel set of rectangular axes passing through a new origin.

Let OX, OY be the original axes and let OX', OY' be new axes parallel to the original through the new origin O'. Produce $Y'O'$ to meet OX in M, so that $OM=a$, $MO'=\beta$.



Let an *arbitrary point* P have co-ordinates (x, y) referred to the original axes OX, OY and co-ordinates (x', y') referred to the new axes $O'X'$, $O'Y'$. Draw PN perp. to OX meeting $O'X'$ in N'. Then

$$ON=x, \quad NP=y, \quad O'N'=x', \quad N'P=y'.$$

$$\therefore \quad x=ON=OM+MN=OM+O'N'=a+x',$$

$$y=NP=NN'+N'P=MO'+N'P=\beta+y'.$$

Thus, to transfer the origin to the point (α, β), the *formula of transformation* is

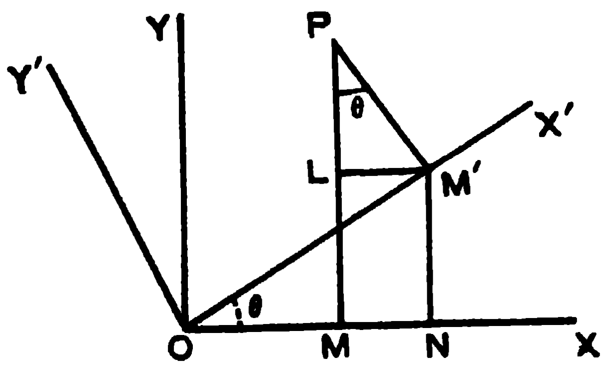
$$\mathbf{x} = \mathbf{x}' + \alpha,$$
$$\mathbf{y} = \mathbf{y}' + \beta.$$

Note. Thus when the origin is transferred to (α, β), the *transformed equation of a curve* is obtained by substituting $x+\alpha$, $y+\beta$ for x, y in the equation of the curve.

4·3. Change of directions of axes.

To change the directions of axes without changing the origin, both systems of co-ordinates being rectangular.

Let OX, OY be the original system of axes and OX', OY' be the new system through the same origin O and let θ be the angle through which the axes are turned so that $\angle XOX' = \angle YOY' = \theta$.



Let the co-ordinates of any arbitrary point P referred to the original axes OX, OY be (x, y) and let those referred to the new axes OX', OY' be (x', y').

Draw PM, PM' perp. to OX, OX' and $M'N$ perp. to OX and $M'L$ perp. to PM.

Then, $OM = x$, $MP = y$, $OM' = x'$, $PM' = y'$,

$$\angle LPM' = 90° - PM'L = \angle LM'O = \angle XOX' = \theta.$$

$$\therefore\ x = OM = ON - MN = ON - LM'$$
$$= OM' \cos\theta - PM' \sin\theta$$
$$= x' \cos\theta - y' \sin\theta.$$

$$y = MP = ML + LP = NM' + LP$$
$$= OM' \sin\theta + PM' \cos\theta$$
$$= x' \sin\theta + y' \cos\theta.$$

Thus, to turn the axes through an angle θ (without change of origin), the *formula of transformation* is

$$\mathbf{x} = \mathbf{x}' \cos\theta - \mathbf{y}' \sin\theta,$$
$$\mathbf{y} = \mathbf{x}' \sin\theta + \mathbf{y}' \cos\theta.$$

Cor. When it is required to change the origin to (α, β) as well as to turn the directions of axes through an angle θ, the formula of transformation will obviously be the combination of the above two formulæ *i.e.*

$$x = \alpha + x' \cos\theta - y' \sin\theta,$$
$$y = \beta + x' \sin\theta + y' \cos\theta.$$

Note 1. This kind of transformation where both the systems of axes are rectangular is called *orthogonal transformation.*

Note 2. Thus when the directions of axes are turned through an angle θ, the *transformed equation of a curve* is obtained by substituting $x \cos\theta - y \sin\theta$ and $x \sin\theta + y \cos\theta$ for x and y in the equation of the curve.

4·4. Illustrative Examples.

Ex. 1. *If the quantity* $ax^2 + 2hxy + by^2$ *becomes* $a'x'^2 + 2h'x'y' + b'y'^2$ *by orthogonal transformation without change of origin,* prove that

$$a + b = a' + b' \text{ and } ab - h^2 = a'b' - h'^2.$$

By applying the formula for orthogonal transformation

$ax^2 + 2hxy + by^2$ becomes

$$= a(x' \cos\theta - y' \sin\theta)^2 + 2h\,(x' \cos\theta - y' \sin\theta) \times (x' \sin\theta + y' \cos\theta) + b(x' \sin\theta + y' \cos\theta)^2$$

$$= a'x'^2 + 2h'x'y' + b'y'^2, \text{ say}$$

where $a' = a \cos^2\theta + 2h \cos\theta \sin\theta + b \sin^2\theta \quad \cdots \quad (1),$

$b' = a \sin^2\theta - 2h \sin\theta \cos\theta + b \cos^2\theta \quad \cdots \quad (2),$

$h' = h\,(\cos^2\theta - \sin^2\theta) - (a - b) \cos\theta \sin\theta \quad \cdots \quad (3).$

Adding (1) and (2), we get $a' + b' = a + b$

$$2a' = a\,(1 + \cos 2\theta) + 2h \sin 2\theta + b(1 - \cos 2\theta)$$
$$= (a + b) + \{2h \sin 2\theta + (a - b) \cos 2\theta\}. \quad \cdots \quad (4)$$

Similarly $2b' = (a + b) - \{2h \sin 2\theta + (a - b) \cos 2\theta\}, \quad \ldots \quad (5)$

$$2h' = 2h \cos 2\theta - (a - b) \sin 2\theta.$$

From (4), (5) and (6),

$$4(a'b'-h'^2) = (a+b)^2 - \{2h \sin 2\theta + (a-b)\cos 2\theta\}^2 - \{2h \cos 2\theta - (a-b)\sin 2\theta\}^2$$
$$= 4(ab-h^2).$$

Hence the result.

Ex 2. *Find the angle through which the axes must be turned so that the expression $ax^2+2hxy+by^2$ may become an expression in which the term xy is absent.*

Let θ be the reqd. angle.

Then substituting $x'\cos\theta - y'\sin\theta$ for x and $x'\sin\theta + y'\cos\theta$ for y in the given expression, as above, we must have h' *i.e.* coefficient of $2xy=0$,

i.e. $h(\cos^2\theta - \sin^2\theta) - (a-b)\cos\theta\sin\theta = 0$,

i.e. $2h\cos 2\theta - (a-b)\sin 2\theta = \theta$,

i.e. $\tan 2\theta = \dfrac{2h}{a-b}$.

$\therefore\ \theta = \frac{1}{2}\tan^{-1}\left(\dfrac{2h}{a-b}\right)$.

Examples IV

1. Transform to parallel axes through the new origin $(2, -4)$ the equations of the following curves

(i) $2x+3y+8=0$;

(ii) $x^2+y^2-4x+8y-17=0$;

(iii) $3y^2-7x+24y+62=0$;

(iv) $5x^2-4y^2-20x-32y-60=0$.

2. *Choose a new origin (α, β) (retaining the directions of axes) so that the equation of the curve

$$5xy+y^2+25x-5y-65=0$$

may be converted into $Ax'y'+By'^2=1$.

Also determine the actual values of A and B.

3. The directions of the axes remaining the same, choose a new origin, such that the *new* co-ordinates of the pair of points whose old co-ordinates are $(5, -13)$ and $(-3, 11)$ may be of the form (h, k) and $(-h, -k)$.

4. Show how to change the origin (without altering the directions of axes) such that the two equations *viz.*

$$x^2+y^2-12x-6y+34=0,$$

$$\text{and} \quad x^2+y^2-10x-6y+30=0$$

may be transformed respectively into equations of the form :

$$x^2+y^2+2\lambda x+\nu=0 \quad \text{and} \quad x^2+y^2+2\mu x+\nu=0.$$

Also work out the actual values of λ, μ, ν.

5. In an orthogonal transformation without change of origin, if (x, y) be the co-ordinates of P referred to the old axes and (X, Y) its co-ordinates referred to the new, prove that

$$x^2+y^2=X^2+Y^2.$$

6. Find the angle through which the axes may be turned so that the equation $\sqrt{3}x+y-4=0$ may be reduced to the form $x=k$; also determine the value of k.

7. (i) The equation $x^2-y^2=a^2$ is transformed to $xy=k^2$ by a change of rectangular axes ; find the inclination of the latter axes to the former and the value of k^2.

8. Transform to axes inclined at $45°$ to the original axes the equation $x^2-y^2=a^2$.

[*C. U. 1940*]

9. Verify that when the axes are turned through an angle of $\frac{1}{8}\pi$ (the origin remaining fixed), each of the following equations

(i) $3x^2+2xy+y^2=60$;

(ii) $7x^2+3xy+4y^2=11$

assumes the symbolic form $Ax'^2+By'^2=1$.

10. The equation $3x^2+2xy+3y^2-18x-22y+50=0$ is transformed to $4x^2+2y^2=1$ when referred to rectangular axes through the point $(2, 3)$. Find the inclination of the latter axes to the former.

[*C. U. 1945*]

11. By transforming to parallel axes through a properly chosen point (h, k) prove that the following equation can be reduced to one containing only terms of the second degree : $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$.

[*C. U. 1940*]

12. Prove that the value of $g^2 + f^2$ in the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ remains unaffected by orthogonal transformation without change of origin.

CHAPTER V

PAIR OF STRAIGHT LINES

[IMPROPER CONIC]

5·1. Homogeneous Quadratic Equation.

The homogeneous quadratic equation $ax^2+2hxy+by^2=0$ *always represents a pair of straight lines, real or imaginary, through the origin.*

$$ax^2+2hxy+by^2=0. \qquad \ldots \quad (1)$$

Multiplying this by a, we have

$$a^2x^2+2ahxy+aby^2=0,$$

or $$a^2x^2+2ahxy+h^2y^2-(h^2-ab)y^2=0,$$

or $$\{ax+hy\}^2-\{\sqrt{(h^2-ab)}y\}^2=0,$$

or $$(ax+hy+\sqrt{h^2-ab}y)(ax+hy-\sqrt{h^2-ab}y)=0.$$

Thus, the given equation is equivalent to

$$ax+(h+\sqrt{h^2-ab})y=0, \qquad \ldots \quad (2)$$

and $$ax+(h-\sqrt{h^2-ab})y=0, \qquad \ldots \quad (3)$$

each of which obviously represents a line passing through the origin.

The two lines (2), (3), form the locus of the equation (1); for (1) is satisfied by the co-ordinates of all points which satisfy (2) and (3).

The two lines will be ***real*** and ***different*** if $h^2>ab$, ***real*** and ***coincident*** if $h^2=ab$ and ***imaginary*** if $h^2<ab$.

✓ *Alternative Method.*

Dividing both sides of the equation (1) by x^2 and b, we have,

$$\left(\frac{y}{x}\right)^2+2\frac{h}{b}\frac{y}{x}+\frac{a}{b}=0. \quad \dots \quad (4)$$

Let m_1, m_2 be the roots of this quadratic in $\frac{y}{x}$.

Then (4) must be equivalent to

$$\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)=0. \quad \dots \quad (5)$$

Thus the two lines represented by (4) *i.e.* by (1) are given by

$$\frac{y}{x}-m_1=0, \quad \text{and } \frac{y}{x}-m_2=0,$$

$$i.e. \ y-m_1x=0, \quad \text{and } y-m_2x=0.$$

Note 1. It should be noted that in this case

$$by^2+2hxy+ax^2=b(y-m_1x)(y-m_2x).$$

Note 2. An equation in which the sum of indices of x and y of every term is the same is called a *homogeneous equation* in x and y. If the sum of indices be two, it is called a homogeneous equation of *second degree*, if the sum of indices be three, it is called a homogeneous equation of *third degree* and so on. Thus, $2x^2-3xy+5y^2$, $6x^3-8x^2y+6xy^2+4y^3$ are homogeneous equations of second and third degrees.

Note 3. As above, it can be shown that *a homogeneous equation of degree* 3 *represents three lines through the origin, one of degree* 4, *represents four lines through the origin* and so on.

✓ 5·2. Angle between the lines $ax^2+2hxy+by^2=0$.

$$ax^2+2hxy+by^2=0. \quad \dots \quad (1)$$

Let the separate equations of the lines represented by (1) be

$$y-m_1x=0, \quad \text{and} \quad y-m_2x=0.$$

Then, $by^2+2hxy+ax^2=b(y-m_1x)(y-m_2x)$. ... (3)

$\therefore$ equating coefficients of x^2 and xy, we get

$$m_1 + m_2 = -2\frac{h}{b}, \qquad m_1 m_2 = \frac{a}{b}.$$

Let φ be the angle between the lines (2); then by Art. 3·6 we have

$$\tan\varphi = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{4\frac{h^2}{b^2} - 4\frac{a}{b}}}{1 + \frac{a}{b}} = \frac{2\sqrt{\mathbf{h}^2 - \mathbf{ab}}}{\mathbf{a}+\mathbf{b}}.$$

Cor. 1. If $h^2 = ab$, $\tan\varphi = 0$ and hence $\varphi = 0$. The angle between the lines is thus zero and since the two lines pass through the origin, the lines become coinicident in this case.

$\therefore$ **Condition of coincidence of the lines is $\mathbf{h^2 = ab}$.**

Cor. 2. If $a + b = 0$, $\tan\varphi = \infty$ and hence $\varphi = \frac{1}{2}\pi$ and therefore the lines are perpendicular.

$\therefore$ **Condition for perpendicularity of the lines is $\mathbf{a + b = 0}$.**

Thus, *the two lines represented by* $ax^2 + 2hxy + by^2 = 0$ *are perpendicular if the algebraic sum of the coefficients of of* x^2 *and* y^2 *is zero.*

Note 1. The lines will be real or imaginary according as ϕ is real or imaginary *i.e.* according as $h^2 >$ or $< ab$.

Note 2. The above result can also be established by taking the separate equations of the lines represented by (i) as

$$l_1 x + m_1 y = 0, \qquad l_2 x + m_2 y = 0. \qquad \ldots \qquad \ldots \quad \text{(ii)}$$

$\therefore$ $ax^2 + 2hxy + by^2 = (l_1 x + m_1 y)(l_2 x + m_2 y)$.

$\therefore$ Comparing coefficients,

$$ll_1 = a, \quad m_1 m_2 = b, \quad l_1 m_2 + l_2 m_1 = 2h.$$

If ϕ be the angle between the lines (ii), then by Art. 3·6

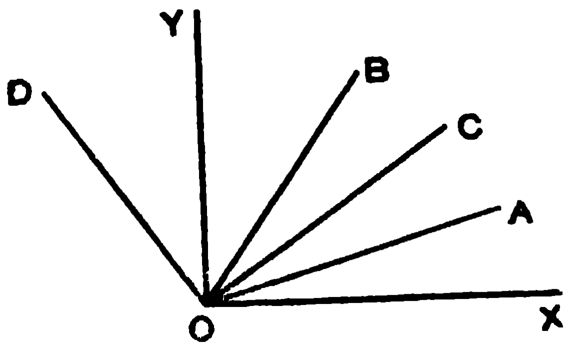
$$\tan\phi = \frac{l_2 m_1 - l_1 m_2}{l_1 l_2 + m_1 m_2}.$$

Now, $(l_2m_1 - l_1m_2)^2 = (l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2$
$= 4(h^2 - ab).$

$\therefore \quad l_2m_1 - l_1m_2 = 2\sqrt{h^2 - ab}$

and $\quad l_1l_2 = a, \quad m_1m_2 = b.$ Hence the result.

5·3. Bisectors of angles between $ax^2 + 2hxy + by^2 = 0$.



Let $ax^2 + 2hxy + by^2 = 0$ represent the lines OA, OB and let their separate equations be $y - m_1x = 0$, $y - m_2x = 0$.

Then

$$y^2 + 2\frac{h}{b}xy + \frac{a}{b}x^2 = (y - m_1x)(y - m_2x).$$

$$\therefore \; m_1 + m_2 = -2\frac{h}{b}, \; m_1m_2 = \frac{a}{b}$$

Let OC, OD be the internal and external bisectors of $\angle AOB$.

Let $\angle AOX = \theta_1$, $\angle BOX = \theta_2$; then $\tan\theta_1 = m_1$, $\tan\theta_2 = m_2$.

Now $\angle XOC = \angle XOA + \angle AOC = \angle XOA + \frac{1}{2}\angle AOB$
$= \angle XOA + \frac{1}{2}(\angle XOB - \angle XOA)$
$= \frac{1}{2}(XOA + \angle XOB = \frac{1}{2}(\theta_1 + \theta_2)$

$\angle XOD = \angle XOC + \angle COD = \frac{1}{2}\pi + \frac{1}{2}(\theta_1 + \theta_2)$, since $\angle COD$ is a right angle.

Hence if θ be the angle which any bisector makes with the x-axis *i.e.* if $\theta = \angle XOC$ or $\angle XOD$, then in either case

$$\tan 2\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = \frac{m_1 + m_2}{1 - m_1m_2}$$

$$= \frac{2h}{a - b}. \quad \ldots \quad \ldots \quad (1)$$

Let (x, y) be the co-ordinates of any point P on either bisector ; then $\tan\theta = \dfrac{y}{x}$.

$$\therefore \quad \text{from (i)}, \; \frac{2h}{a-b} = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2y/x}{1-y^2/x^2}$$

$$= \frac{2xy}{x^2-y^2}.$$

$$\therefore \quad \frac{\mathbf{x^2-y^2}}{\mathbf{a-b}} = \frac{\mathbf{xy}}{\mathbf{h}}.$$

This, being a relation between the co-ordinates of any point on either bisector, is the joint equation of the two bisectors.

Note. It can be easily verified that these bisectors are at right angles.

Alternative Method.

Suppose the separate equations of the lines are

$$y - m_1 x = 0 \quad \text{and} \quad y - m_2 x = 0 \qquad \cdots \quad (1)$$

so that $\quad m_1 + m_2 = -2h/b$ and $m_1 m_2 = a/b \qquad \cdots \quad (2)$

The equations of their bisectors are by Art. 3·16

$$\frac{y-m_1x}{\sqrt{1+m_1^2}} = \pm \frac{y-m_2x}{\sqrt{1+m_2^2}}.$$

$\therefore$ their joint equation is

$$\left\{\frac{y-m_1x}{\sqrt{1+m_1^2}} - \frac{y-m_2x}{\sqrt{1+m_1^2}}\right\}\left\{\frac{y-m_1x}{\sqrt{1+m_1^2}} + \frac{y-m_2x}{\sqrt{1+m_2^2}}\right\} = 0,$$

or
$$\frac{(y-m_1x)^2}{1+m_1^2} - \frac{(y-m_2x)^2}{1+m_2^2} = 0,$$

or
$$(1+m_2^2)(y^2 - 2m_1xy + m_1^2x^2) - (1+m_1^2)(y^2 - 2m_2xy + m_2^2x^2) = 0,$$

or $\quad (m_1+m_2)(x^2-y^2)+2(m_1m_2-1)(m_1-m_2)xy=0,$

or $\quad (m_1+m_2)(x^2-y^2)+2(m_1m_2-1)xy=0$, since $m_1 \neq m_2$

or $\quad -\dfrac{2h}{b}(x^2-y^2)+2\left(\dfrac{a}{b}-1\right)xy=0$ by (2),

or $$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

5·4. Condition for a pair of lines.

To find the condition that the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \dots \quad (1)$$

may represent two straight lines.

If the left side of the above equation breaks up into two factors each of the first degree in x, y, then it will represent two lines.

Treating the equation as a quadratic in x and supposing $a \neq 0$, we have

$$ax^2+2(hy+g)x+(by^2+2fy+c)=0.$$

$$\therefore \quad x=\frac{-2(hy+g)\pm\sqrt{4(hy+g)^2-4a(by^2+2fy+c)}}{2a}.$$

Cancelling 2 from numerator and denominator, multiplying cross-wise and transposing, we get

$$\begin{aligned} ax+hy+g &= \pm\sqrt{(hy+g)^2-a(by^2+2fy+c)} \\ &= \pm\sqrt{y^2(h^2-ab)+2y(gh-af)+g^2-ac} \qquad \cdots \quad (2) \end{aligned}$$

Unless the right-side of (2) is the form $\pm(ly+m)$, the left-side of the given equation cannot be resolved into two factors. The condition for this is that the expression under radical in (2) must be a perfect square *i.e.*

$$4(gh-af)^2=4(h^2-ab)(g^2-ac),$$

i.e. $\quad g^2h^2-2afgh+a^2f^2=g^2h^2-abg^2-ach^2+a^2bc,$

i.e. $$\mathbf{abc+2fgh-af^2-bg^2-ch^2=0.} \qquad \cdots \quad (A)$$

The above condition can be written in the form

$$\Delta \equiv \begin{vmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{h} & \mathbf{b} & \mathbf{f} \\ \mathbf{g} & \mathbf{f} & \mathbf{c} \end{vmatrix} = 0. \qquad \cdots \quad (B)$$

Note 1. The **above condition is both necessary and sufficient.** We have proved that the condition is necessary. Now if we start with the relation (A), we find that the exp. under the radical on the right side of (2) is a perfect square *i.e.* is of the form $\pm(ly+m)$. Therefore left side of (1) is resolvable into two linear factors and hence equation (1) represents two lines. Thus the condition is sufficient.

Note 2. In the above proof we have assumed that $a \neq 0$ and have solved the equation for x. If $a=0$ and $b \neq 0$, we shall solve it for y and proceed exactly in the same way and get the condition $2fgh-bg^2-ch^2=0$, which is in fact the same as condition (A), because here $a=0$. If however $a=b=0$, then we proceed as follows.

The given equation (1) being divided by $2h$, takes the form

$$xy+\frac{g}{h}x+\frac{f}{h}y+\frac{c}{2h}=0,$$

i.e.
$$\left(x+\frac{f}{h}\right)\left(y+\frac{g}{h}\right)=\frac{fg}{h^2}-\frac{c}{2h}=\frac{2fgh-ch^2}{2h^2}.$$

This evidently can represent a pair of lines when and only when the right side is zero *i.e.* $2fgh-ch^2=0$, which evidently is the same as the condition (A), because here $a=b=0$.

Note 3. The quantity on the left side of (A) is called the *Discriminant* of the given equation.

5·5. On the lines represented by $\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$.

Suppose
$$ax^2+2hxy+by^2+2gx+2fy+c = (lx+my+n)(l'x+m'y+n') \qquad \cdots \quad (1)$$

then
$$ll'=a,\ mm'=b,\ nn'=c,\ lm'+l'm=2h,$$
$$mn'+m'n=2f,\ nl'+n'l=2g. \qquad \cdots \quad (2)$$

(i) Angle between the lines.

If φ be the angle between these lines

$$\tan\varphi = \frac{lm' - l'm}{ll' + mm'} = \frac{\sqrt{(lm' + l'm)^2 - 4ll'mm'}}{ll' + mm'}$$

$$= \frac{2\sqrt{h^2 - ab}}{a+b}. \quad \cdots \quad \cdots \quad (3)$$

Since angle between the lines $ax^2 + 2hxy + by^2 = 0$ is also the same as (3), [Art. 5·2], it follows, *the lines represented by* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *are parallel to the lines*

$$ax^2 + 2hxy + by^2 = 0.$$

(ii) Condition for perpendicularity of the lines.

The lines will be perp. if $ll' + mm' = 0$, *i.e.* if $a + b = 0$. This also follows from (3). Hence *the given equation represents a pair of perpendicular lines if* $\Delta = 0$ *and* $a + b = 0$.

(iii) Condition for parallelism of the lines.

The lines will be parallel if $\frac{l}{l'} = \frac{m}{m'}$, *i.e.* $lm' - l'm = 0$,

i.e. if $(lm' + l'm)^2 - 2ll'mm' = 0$, if $h^2 - ab = 0$. This also follows from (3). Hence *the given equation represents a pair of parallel lines if* $\Delta = 0$ *and* $h^2 = ab$.

(v) Condition for coincidence of the lines.

The lines will be coincident if $\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'}$,

i.e. if $lm' - l'm = 0, \quad mn' - m'n = 0, \quad nl' - n'l = 0,$

i.e. if $h^2 - ab = 0, \quad f^2 - bc = 0, \quad g^2 - ac = 0.$

In such a case, the given equation becomes a perfect square. Hence *the given equation will represent a pair of coincident lines if*

$$\Delta = 0 \text{ and } f^2 - bc = 0,\ g^2 - ca = 0,\ h^2 - ab = 0.$$

(vi) Point of intersection of the lines.

Let (α, β) be the point of intersection of the lines. Transferring the origin to (α, β), keeping the directions of axes unchanged, the given equation by Art. 4·2 becomes

$$a(x+\alpha)^2 + 2h(x+\alpha)(y+\beta) + b(y+\beta)^2$$
$$+ 2g(x+\alpha) + 2f(y+\beta) + c = 0,$$

or $$ax^2 + 2hxy + by^2 + 2x(a\alpha + h\beta + g) + 2y(h\alpha + b\beta + f)$$
$$+ a\alpha^2 + 2h\alpha\beta + \cdots\cdots\cdots\cdots = 0 \quad \cdots \quad (1)$$

Since (1) now represents a pair of lines through the origin the left side must be a homogeneous quadratic in x, y, *i.e.* must be reduced to the form $ax^2 + 2hxy + by^2 = 0$, which requires that the coefficients of x, y and the constant term be zero.

$$\therefore \qquad a\alpha + h\beta + g = 0$$
$$h\alpha + b\beta + f = 0,$$

from which we get

$$\frac{\alpha}{hf - bg} = \frac{\beta}{gh - af} = \frac{1}{ab - h^2}.$$

$$\therefore \qquad \alpha = \frac{hf - bg}{ab - h^2}, \; \beta = \frac{gh - af}{ab - h^2},$$

or, in the notation of co-factors (*Art. 1·7*),

$$\alpha = \frac{G}{C}, \; \beta = \frac{F}{C}.$$

Alternative Method.

Suppose $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$
$$\equiv (lx + my + n)(l'x + m'y + n'). \quad \ldots \quad (1)$$

This relation being true for all values of x and y, it must admit of partial differentiation w. r. t. each of the two variables x and y. So we have

$$2(ax + hy + g) = l'(lx + my + n) + l(l'x + m'y + n'), \quad \ldots \quad (2)$$

and $$2(hx + by + f) = m'(lx + my + n) + m(l'x + m'y + n'). \quad \ldots \quad (3)$$

Let (α, β) be the point of intersection of the lines represented by the general equation.

$$\therefore \quad l\alpha+m\beta+n=0, \; l'\alpha+m'\beta+n'=0.$$

Substituting α, β for x, y in (2), (3), we get

$$a\alpha+h\beta+g=0 \qquad \ldots \qquad \ldots \qquad (5)$$

$$h\alpha+b\beta+f=0. \qquad \ldots \qquad \ldots \qquad (6)$$

From these two equations we get (α, β) as before.

Note. From above we see that if $f(x, y)=0$ be the equation of two lines, the point of intersection would be obtained by solving the equations $\frac{\delta f}{\delta \alpha}=0, \frac{\delta f}{\delta \beta}=0.$

(vii) **Bisectors of the lines.**

Let (α, β) be the point of intersection of the given lines. Referred to parallel axes through (α, β), as origin, the equation of the given lines reduces to the homogeneous form

$$ax^2+2hxy+by^2=0$$

of which the bisectors are given by

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

Reverting now to the old axes, this equation becomes

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h}.$$

which is the required equation of the bisectors of the given lines.

Note. It should be noted that with reference to the new origin, the co-ordinates of the old origin are $(-\alpha, -\beta)$.

5·6. A special pair of lines.

To find the equation of the lines joining the origin to the points of intersection of the curve

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad (1)$$

with the line $\qquad lx+my+n=0 \quad (2).$

Make the first equation homogeneous with the help of the second equation written in the form

$$\frac{lx+my}{n}=-1$$

and we get

$$ax^2+2hxy+by^2-2(gx+fy).\ \frac{lx+my}{n}+c\left(\frac{lx+my}{n}\right)^2=0 \qquad (3)$$

as the reqd. equation.

On simplification, this equation assumes the form

$$Px^2+2Qxy+Ry^2=0.$$

The equation (3) being a homogeneous second degree equation represents a pair of lines through the origin. Moreover, it is satisfied by the co-ordinates of the points whose co-ordinates satisfy equations (1) and (2) *i.e.* of the points where the line (2) cuts the curve (1).

5·7. Illustrative Examples.

Ex. 1. *Show that the equation*

$$2x^2-xy-y^2-10x+7y+8=0$$

represents two straight lines; find their point of intersection and the angle between them.

The given equation can be written as

$$2x^2-x(y+10)-(y^2-7y-8)=0.$$

Solving it as a quadratic in x, we get

$$x=\frac{(y+10)\pm\sqrt{(y+10)^2+8(y^2-7y-8)}}{4}$$

$$=\frac{y+10\pm(3y-6)}{4}=y+1 \text{ or } \frac{-y+8}{2}.$$

Hence the factors of the left side of the equation are

$$x-y-1 \text{ and } 2x+y-8.$$

$\therefore$ the given equation represents the two straight lines $x-y-1=0$ and $2x+y-8=0$.

Solving $x-y-1=0$ and $2x+y-8=0$, we easily get $x=3$, $y=2$ for the point of intersection.

By Art. 3·6, the angle between them is $\tan^{-1} 3$.

Ex. 2. *Find the values of* λ *so that the equation*

$$12x^2+\lambda xy+2y^2+11x-5y+2=0$$

may represent two lines. [*C. U. 1939*]

Comparing the given equation with the general equation of the second degree, we have

$$a=12, \quad b=2, \quad c=2, \quad h=\tfrac{1}{2}\lambda, \quad f=-\tfrac{5}{2}, \quad g=\tfrac{11}{2}.$$

Now, applying the criterion $abc+2fgh-af^2-bg^2-ch^2=0$,

we have $48-\frac{55}{4}\lambda-75-\frac{121}{2}-\frac{h^2}{2}=0$,

or $2\lambda^2+55\lambda+350=0$. $\therefore (\lambda+10)(2\lambda+35)=0$. $\therefore \lambda=-10, -\frac{35}{2}$.

Ex. 3. *Find the condition that one of the lines* $ax^2+2hxy+by^2=0$ *may conicide with one of the lines* $a'x^2+2h'xy+b'y^2=0$.

The given equations can be written as

$$b\left(\frac{y}{x}\right)^2+2h\left(\frac{y}{x}\right)+a=0, \;\ldots\; (1) \quad b'\left(\frac{y}{x}\right)^2+2h'\left(\frac{y}{x}\right)+a'=0. \;\ldots\; (2)$$

Let $y-mx=0$ be the equation of the line which is common to both pairs; then $y-mx$ must be a factor of the left side of (1) and (2) and hence $m\ (=y/x)$ must be a root of both the equations (1) and (2).

$$\therefore \quad bm^2+2hm+a=0 \quad \ldots \quad \ldots \quad (3)$$

$$\therefore \quad b'm^2+2h'm+a'=0. \quad \ldots \quad \ldots \quad (4)$$

From (3) and (4), by the rule of Cross-Multiplication,

$$\frac{m^2}{2(ha'-h'a)}=\frac{m}{ab'-a'b}=\frac{1}{2(bh'-b'h)}.$$

$$\therefore \quad \frac{ha'-h'a}{bh'-b'h}=m^2=\left\{\frac{ab'-a'b}{2(bh'-b'h)}\right\}^2$$

Hence the reqd. condition is

$$(ab'-a'b)^2=4(ha'-h'a)(bh'-b'h).$$

Ex. 4. *Prove that the equation*

$$m(x^3-3xy^2)+y^3-3x^2y=0$$

represents three lines equally inclined to one another.

Let $y=x\tan\theta$ be any line of the system, so that θ is the inclination of the line to the x-axis. Substituting $x\tan\theta$ for y in the given equation, we have

$$m(1-3\tan^2\theta)+\tan^3\theta-3\tan\theta=0 \qquad \ldots \quad (1)$$

which is a cubic equation in $\tan\theta$, giving the inclination of the three lines to the x-axis.

From (1), we get $m=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=\tan 3\theta$.

Let $m=\tan 3\alpha$, where α has the smallest possible positive value.

Then, $\quad \tan 3\theta=\tan 3\alpha$

$\therefore \quad 3\theta=n\pi+3\alpha, \quad$ or $\quad \theta=\frac{1}{3}n\pi+\alpha.$

Giving n the values 0, 2, 4, in succession, we have

$$\theta=\alpha,\ 120^\circ+\alpha,\ 240^\circ+\alpha$$

which give the inclinations of the three lines to the x-axis, their mutual inclinations being 120°.

Hence these lines are equally inclined to each other.

Note. It should be noted that for any other value of n, no new line will be obtained.

Examples V

1. Find the angle between the following pair of lines

(i) $3x^2-14xy-5y^2=0$;

(ii) $x^2-2xy\sec\alpha+y^2=0$;

(iii) $x^2-2xy\cot\theta-y^2=0$.

2. Find the equations of the bisectors of the angles between the lines

(i) $(x^2+y^2)\sin\alpha+2xy\cot\alpha=0$;

(ii) $a^2(x+y)^2+b^2(x-y)^2+c^2(x^2-y^2)=0$.

3. Prove that the lines

$$y^2-4xy-x^2=0 \text{ and } y^2+xy-x^2=0,$$

bisect the angles between one another.

4. Show that the equation of the lines bisecting the angles between the bisectors of the pair of lines

$$ax^2 + 2hxy + by^2 = 0$$

is
$$(a - b)(x^2 - y^2) + 4hxy = 0.$$

5. Show that the angle between one of the lines

$$ax^2 + 2hxy + by^2 = 0$$

and one of the lines

$$(a - \lambda)x^2 + 2hxy + (b - \lambda)y^2 = 0$$

is equal to the angle between the other two lines of the system. [*C. U. 1934*]

6. If the pair of lines

$$x^2 - 2axy - y^2 = 0 \text{ and } x^2 - 2bxy - y^2 = 0$$

be such that each pair bisects the angle between the other pair, prove that $ab + 1 = 0$.

7. Show that the pair of lines

$$p^2x^2 + 2r(p + q)xy + q^2y^2 = 0$$

is equally inclined to the pair

$$px^2 + 2rxy + qy^2 = 0.$$

8. Show that the product of the perpendiculars from (x', y') on the lines $ax^2 + 2hxy + by^2 = 0$

is
$$\frac{ax'^2 + 2hx'y' + by'^2}{\sqrt{(a - b)^2 + h^2}}.$$

9. Show that the pair of lines $ax^2 + 2hxy + by^2 = 0$ is perpendicular to the pair $bx^2 - 2hxy + ay^2 = 0$.
[*C. U. 1935*]

10. Find the equation of two lines through the origin perpendicular to the lines

$$5x^2 - 7xy - 3y^2 = 0.$$

11. Find the condition that one of the lines

$$ax^2 + 2hxy + by^2 = 0$$

may be perpendicular to one of the lines

$$a'x^2 + 2h'xy + b'y^2 = 0.$$

12. Show that the triangle formed by the lines

$$ax^2 + 2hxy + by^2 = 0 \text{ and } lx + my = 1$$

is right-angled if $(a+b)(al^2 + 2hlm + bm^2) = 0$.

13. (i) Find the area of the triangle formed by the lines

$$2x - y - 6 = 0 \text{ and } 3x^2 - 4xy + y^2 = 0.$$

(ii) Show that the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$, and $lx + my = 1$,

is $$\frac{\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}.$$

14. Prove that the line $ax + by + c = 0$ and the two lines $(ax+by)^2 - 3(ay - bx)^2 = 0$ form the sides of an equilateral triangle.

15. Show that $y^3 - x^3 + 3xy(y - x) = 0$ represents three lines equally inclined to one another. [*C. U. 1940*]

[*The lines are* $y = x$, $y = (-2 + \sqrt{3})x$, $y = (-2 - \sqrt{3})x$; *show that the obtuse angle between each pair is* 120°.]

16. Show that each of the following equations represents two lines ; find also thier point of intersection and the angle between them :

(i) $6x^2 - 5xy + y^2 + 17x - 7y + 12 = 0$.

(ii) $x^2 - y^2 - 2x + 6y - 8 = 0$.

(iii) $2x^2 + 3xy + y^2 + 5x + 2y - 3 = 0$.

(iv) $2y^2 + 3xy - 5y - 6x + 2 = 0$.

17. Find the value of λ so that the following equations may represent two lines

(i) $2x^2 + 3xy - 2y^2 + 7x + y + \lambda = 0$.

(ii) $x^2 + \lambda xy - 2y^2 + 3y - 1 = 0$.

18. (i) Prove that $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ represents two parallel lines. [*C. U. 1937, 1944*]

(ii) Show that $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ represents a pair of perpendicular lines. [*C. U. 1943*]

(iii) Prove that $4x^2-4xy+y^2+8x-4y+3=0$ represents a pair of parallel lines and find the distance between them.

(iv) Show that $x^2-4xy+4y^2+2x-4y+1=0$ represents a pair of coincident lines.

19. Show that

$$(ab-h^2)(ax^2+2hxy+by^2+2gx+2fy)+af^2+bg^2-2fgh$$

$=0$ represents a pair of straight lines.

20. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two intersecting lines, the distance of their point of intersection from the origin is $\sqrt{\frac{A+B}{C}}$, where $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$.

21. Show that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two parallel lines if $\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$ and that when these conditions are satisfied, the distance between them is $2\sqrt{\left\{\frac{g^2-ac}{a(a+b)}\right\}}$.

22. Show that $x^2+4xy+4y^2-5x-10y+4=0$ and $4x^2-4xy+y^2+4x-2y-3=0$ represent straight lines which form a parallelogram.

23. Show that the four lines represented by

$$(y-mx)^2=a^2(1+m^2)$$
$$(y-nx)^2=a^2(1+n^2)$$

form a rhombus.

24. Show that the 4 lines given by $12x^2+7xy-12y^2=0$ and $12x^2+7xy-12y^2-x+7y-1=0$ form the sides of a square.

25. Find the equations of the bisectors of the angles between the two lines $6x^2+7xy-5y^2-22x-41y-8=0$.

26. Prove that the pair of lines $6x^2 + xy - 12y^2 - 14x + 47y - 40 = 0$ are equally inclined to the pair

$$14x^2 + xy - 4y^2 - 30x + 15y = 0.$$

27. Prove that the angle between the lines joining the origin to the intersections of the line $y = 3x + 2$ with the curve

$$x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0 \quad \text{is } \tan^{-1}\frac{2\sqrt{2}}{3}.$$

[*C. U. 1945*]

28. Find the condition that the lines joining the origin to the points where the line $lx + my = n$ meets the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ may be at right angles.

[*C. U. 1935*]

29. Find the value of k so that the lines which join the origin to the points of intersection of the line $y - x - k = 0$ and the curve $x^2 + y^2 + 4x - 6y - 36 = 0$ may be at right angles.

30. Show that there are two lines joining the origin to the points of intersection of the curves,

$$ax^2 + 2hxy + by^2 + 2gx = 0$$

and $$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$$

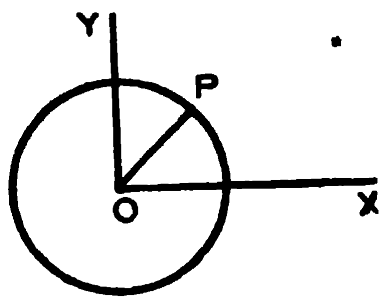
and they will be at right angles if $\dfrac{a+b}{g} = \dfrac{a'+b'}{g'}$.

CHAPTER VI

CIRCLE

6·1. Standard Equation.

To find the equation of a circle whose radius is a and whose centre is taken as the origin of co-ordinate axes.



Let (x, y) be the co-ordinates of any point P on the circle.

Then, $\qquad OP^2 = a^2.$

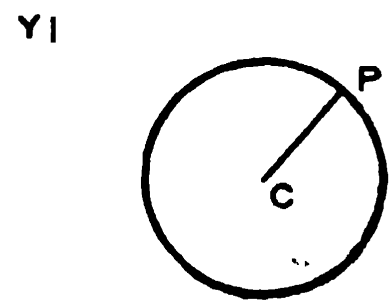
$$\therefore \quad \mathbf{x^2 + y^2 = a^2}.$$

This, being the relation connecting the co-ordinates of any point on the circle, is the equation of the circle.

6·2. General Equation.

To find the equation of a circle whose centre is (α, β) and whose radius is r.

Let $C(\alpha, \beta)$ be the centre of the circle and P be any point on the circle whose co-ordinates are (x, y).



Then, $\quad CP^2 = r^2.$

$\therefore$ by Art. 2·4,

$$(\mathbf{x} - \alpha)^2 + (\mathbf{y} - \beta)^2 = \mathbf{r}^2. \quad \ldots(1)$$

On simplifying the equation (1) reduces to

$$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0,$$

which is of the form

$$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}. \quad \cdots \quad (2)$$

So (2) is very often taken as the general equation of a circle.

6·3. Locus of $x^2+y^2+2gx+2fy+c=0$.

To prove that the equation

$$x^2+y^2+2gx+2fy+c=0 \qquad \cdots \qquad (1)$$

always represents a circle and to find its centre and radius.

The given equation can be written as

$$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c,$$

or $(x+g)^2+(y+f)^2=g^2+f^2-c, \qquad \cdots \qquad (2)$

or $\{x-(-g)\}^2+\{y-(-f)\}^2=\{\sqrt{(g^2+f^2-c)}\}^2. \quad (3)$

This shows that the equation is the locus of a point such that its distance from the fixed point $(-g, -f)$ is constant, being equal to $\sqrt{g^2+f^2-c}$.

Hence, *the equation represents a circle,*

whose **centre** is $(-\mathbf{g}, -\mathbf{f})$,

and **radius** is $\sqrt{\mathbf{g}^2+\mathbf{f}^2-\mathbf{c}}$.

If $g^2+f^2-c<0$, the radius which $=\sqrt{g^2+f^2-c}$ becomes imaginary. Hence *the condition that the above equation should represent a real circle is* $g^2+f^2-c \nless 0$. Thus we see that the reality of coefficients in the equation of a circle does not necessarily imply the reality of the circle.

Note 1. In the limiting case when $g^2+f^2-c=0$, the radius of the circle being zero, the circle reduces to a point *viz.* the centre. In such a case, the circle is called a **point-circle.** The equation (2) in this case becomes

$$(x+g)^2+(y+f)^2=0$$

which is only satisfied by $(-g, -f)$, the co-ordinates of the centre.

Similarly the standard equation reduces to a point-circle $x^2+y^2=0$ when $a=0$.

Note 2. The three independent constants g, f, c in the general equation of a circle signify that a circle can be found to satisfy only three independent geometrical conditions *viz.* those of passing through 3 points.

6·4. Condition for a circle.

To find the condition that the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \cdots \quad (1)$$

should represent a circle.

Comparing (1) with the general equation of the circle

viz. $\quad x^2+y^2+2gx+2fy+c=0,$

which on being multiplied by any constant k may as well be written in the form

$$kx^2+ky^2+2g'x+2f'y+c'=0$$

(where $g'=gk, f'=fk, c'=ck$).

we conclude that *the general equation of the second degree will represent a circle* if $\mathbf{a=b}$ and $\mathbf{h=0}$,

***i. e.* if the coefficients of x^2 and y^2 be the same and coefficient of xy be zero.**

Cor. Thus, we see that the equation

$$a(x^2+y^2)+2gx+2fy+c=0$$

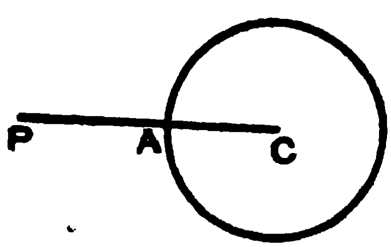
also *always represents a circle.* Its centre and radius can easily be found as in Art. 6·3 by first dividing the equation by a.

6·5. Position of a point in relation to a circle.

The point (x_1, y_1) lies outside, upon or inside the circle $x^2+y^2-a^2=0$ according as

$$x_1^2+y_1^2-a^2 > = \text{ or } < 0.$$

Let P be the point (x_1, y_1) and $C(0, 0)$ the centre.



Then $PC^2=x_1^2+y_1^2$.

Now P lies outside, upon or inside the circle

according as $PC > = <$ its radius,

i.e. „ „ $PC^2 > = <$ (radius)^2,

$x_1^2+y_1^2 > = < a^2$,

i.e. according as $x_1^2+y_1^2-a^2 > =$ or < 0.

Similarly it can be shown that the *point* (x_1, y_1) lies *outside, upon or inside the circle*

$$x^2+y^2+2gx+2fy+c=0$$

according as $x_1{}^2+y_1{}^2+2gx_1+2fy_1+c > \; = \text{ or } <0.$

For, in this case, $PC^2=(x_1+g)^2+(y_1+f)^2$,

and $(\text{radius})^2=g^2+f^2-c$.

6·6. Intersection of a line and a circle.

To find the points of intersection of the line

$$y=mx+c \qquad \dots \qquad \dots \quad (1)$$

with the circle $x^2+y^2=a^2$. $\quad \dots \quad \dots \quad (2)$

The abscissæ of the common points of intersection will be the roots of the quadratic in x obtained by eliminating y between the equations (1) and (2).

Substituting the value of y from (1) in (2), we get

$$x^2+(mx+c)^2=a^2,$$

$$\text{or} \qquad (1+m^2)x^2+2mcx+c^2-a^2=0. \qquad \dots \quad (3)$$

Similarly, eliminating x between (1) and (2), the quadratic in y giving the ordinates of the common points is found to be

$$\left(\frac{y-c}{m}\right)^2+y^2=a^2,$$

$$\text{or} \qquad (1+m^2)y^2-2cy+c^2-a^2m^2=0. \qquad \dots \quad (4)$$

From (3), it is clear that *a st. line intersects a circle in two points, real, coincident* or *imaginary*, according as the roots of (3) are real, coincident, or imaginary,

i.e. according as

$(2mc)^2-4(1+m^2)(c^2-a^2)$ is positive, zero or neg.

i.e. according as

$a^2(1+m^2)-c^2$ is positive, zero or negative,

i.e. according as $\quad c^2 < \; = \text{or} > a^2(1+m^2)$.

,, numerically, $c < \; = \; > a\sqrt{1+m^2}$

,, $\dfrac{c}{\sqrt{1+m^2}} < \; = \; > a,$

i.e., according as the length of the perpendicular from the centre upon the line is numerically less than, equal to, or greater than the radius of the circle.

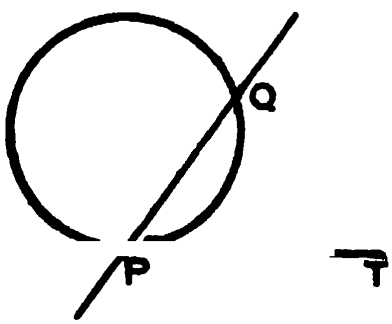
Note. The points of intersection of the line $lx+my+n=0$ and the circle $x^2+y^2+2gx+2fy+c=0$ can be obtained exactly in the same way.

6·7. Tangent at a point.

To find the equation of the tangent at the point (x_1, y_1) to the circle

(i) $x^2+y^2=a^2$;

(ii) $x^2+y^2+2gx+2fy+c=0$.



Let P be the given point (x_1, y_1) and let $Q(x_2, y_2)$ be a point on the circle very close to P.

The equation to PQ is

$$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1). \quad \ldots \ (1)$$

(i) Since (x_1, y_1) (x_2, y_2) lie on the circle $x^2+y^2=a^2$,

$\therefore \quad x_1{}^2+y_1{}^2=a^2=x_2{}^2+y_2{}^2$,

or, $\quad x_1{}^2-x_2{}^2+y_1{}^2-y_2{}^2=0$,

or, $(x_1-x_2)(x_1+x_2)=-(y_1-y_2)(y_1+y_2)$.

$$\therefore \quad \frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2}{y_1+y_2}. \quad \ldots \quad (2)$$

By (2), the equation (1) becomes

$$y-y_1=-\frac{x_1+x_2}{y_1+y_2}(x-x_1),$$

or, $\quad (x-x_1)(x_1+x_2)+(y-y_1)(y_1+y_2)=0, \quad \ldots \quad (3)$

which is the *equation of the chord PQ.*

Let Q approach very close to P, so that ultimately Q coincides with P; then $x_2 = x_1$, $y_2 = y_1$ and the secant PQ becomes the tangent at P.

$\therefore$ the *equation of the tangent at P is*

$$(x - x_1)2x_1 + (y - y_1)2y_1 = 0,$$

i.e., $xx_1 + yy_1 = x_1^2 + y_1^2 = a^2$,

$$\textit{i.e.}\quad \mathbf{xx_1 + yy_1 = a^2}.$$

(ii) Since (x_1, y_1), (x_2, y_2) lie on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$,

we have $\left.\begin{array}{l} x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \\ x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0. \end{array}\right\}$ ... (4)

By subtraction,

$$x_1^2 - x_2^2 + y_1^2 - y_2^2 + 2g(x_1 - x_2) + 2f(y_1 - y_2) = 0,$$

or $(x_1 - x_2)(x_1 + x_2 + 2g) + (y_1 - y_2)(y_1 + y_2 + 2f) = 0.$

$$\therefore \quad \frac{y_1 - y_2}{x_1 - x_2} = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}. \quad .. \quad (5)$$

Hence the equation (1) becomes

$$y - y_1 = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}(x - x_1)$$

or, $(x - x_1)(x_1 + x_2 + 2g) + (y - y_1)(y_1 + y_2 + 2f) = 0$
which is the *equation of the chord PQ*. .. (6)

As before, putting $x_2 = x_1$, $y_2 = y_1$ in (6) *the equation of the tangent at P* is

$$(x - x_1)(2x_1 + 2g) + (y - y_1)(2y_1 + 2f) = 0,$$

or, $x(x_1 + g) + y(y_1 + f) = x_1^2 + y_1^2 + gx_1 + fy_1$

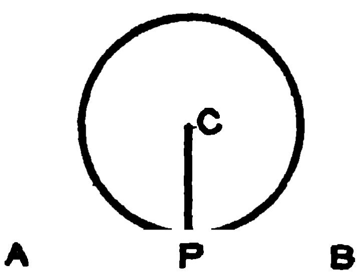
$$= -(gx_1 + fy_1 + c) \text{ by } (4)$$

or, $\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.}$ .. (7)

Note. As an aid to memory, the following points should be noted. Replace the terms x^2, y^2, $2x$, $2y$ in the equation of the circle (ii) by xx_1, yy_1, $x+x_1$, $y+y_1$ respectively and we get the equation of the tangent at (x_1, y_1) to the circle (ii). This rule is also applicable in the case of equations of tangents to parabola, ellipse and hyperbola.

Alternative Method.

The equation of the tangent can also be obtained by using the property that the tangent at any point of a circle is perp. to the line joining the point of contact to the centre of the circle.



Any line through $P(x_1, y_1)$ is

$$y - y_1 = m(x - x_1). \quad \cdots \quad (1)$$

For the circle (i), centre C is (0, 0).

equation of PC is $y = \frac{y_1}{x_1}x$. ...(2)

If (1) be a tangent to the circle (i), it must be perp. to (2) *i.e.*, $m \times \frac{y_1}{x_1} = -1$, *i.e.*, $m = -\frac{x_1}{y_1}$..

Substituting this value of m in (1) and simplifying, the reqd. equation of the tangent is obtained.

For the circle (ii), centre C is $(-g, -f)$.

$\therefore$ Equation of PC is $y - y_1 = \frac{y_1+f}{x_1+g}(x - x_1)$. $\cdots$ (3)

If (1) be a tangent to the circle (ii), it must be perp. to (3), *i.e.*, $m \times \frac{y_1+f}{x_1+g} = -1$, *i.e.*, $m = -\frac{x_1+g}{y_1+f}$.

Substituting this value of in (3), we get the reqd. equation of the tangent

$$y - y_1 = -\frac{x_1+g}{y_1+f}(x - x_1),$$

or $\quad (x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0,$

which can be simplified as before.

6·8. Condition of tangency of a line.

(A) *To find the condition that* $y = mx + c$ ... (1) *shall touch the circle* $x^2 + y^2 = a^2$ ... (2).

Substituting the value of y from (1) in (2), the abscissæ of the points of intersection of (1) and (2) are given by

$$x^2 + (mx + c)^2 = a^2,$$

$$i.e. \quad (1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0. \qquad \ldots \quad (3)$$

If the line (1) touches the circle (2), the two points of intersection of (1) and (2) become coincident and hence the two roots of (3) become equal.

$$\therefore \qquad 4m^2c^2 = 4(1 + m^2)(c^2 - a^2),$$

$$i.e. \qquad c^2 = a^2(1 + m^2),$$

$$i.e. \qquad \mathbf{c = \pm a\sqrt{1 + m^2}} \qquad \ldots \quad (4)$$

which is therefore the condition of tangency.

Thus each of the **the lines $\mathbf{y = mx \pm a\sqrt{1+m^2}}$ is always a tangent to the circle $\mathbf{x^2 + y^2 = a^2}$**, whatever be the value of m.

The *general equation of a tangent to the circle* $x^2 + y^2 = a^2$ *is representable in the form*

$$y = mx \pm a\sqrt{1 + m^2} \qquad \ldots \quad (5)$$

where m *is a variable parameter.*

Cor. 1. For any particular value of m, the two equations

$$y = mx + a\sqrt{1 + m^2}, \qquad y = mx - a\sqrt{1 + m^2}$$

represent a pair of *parallel* tangents to the circle and the line through the points of contact is $x + my = 0$.

Cor. 2. It can be easily shown directly or by change of origin that each of the lines $y - k = m(x - h) \pm a\sqrt{1 + m^2}$ is always a tangent to the circle $(x - h)^2 + (y - k)^2 = a^2$.

Note. The above condition might have been obtained also from the equation $(1+m^2)y^2 - 2cy + c^2 - a^2m^2 = 0$, which gives the ordinates of the points of intersection. Each of the equal roots of (3) and that of this equation give respectively the abscissa and ordinate of the *point of contact* of the line when condition (4) is fulfilled.

Alternative Method.

When a line touches a circle, the perp. from the centre upon the line is equal to the radius. Since the centre of the ⊙ is (0, 0), the condition that (1) shall touch (2) is

$$\frac{}{\sqrt{1+m^2}} \quad \pm a.$$

Hence etc.

(B) *To find the condition that the line* $lx+my+n=0$ ... (6) *shall touch the circle* $x^2+y^2+2gx+2fy+c=0$ ... (7).

Since the centre of ⊙ is $(-g, -f)$ and radius is $=\sqrt{g^2+f^2-c}$, the condition of tangency of line (6) with the circle (7) is

$$\frac{-lg-mf+n}{\sqrt{l^2+m^2}} = \pm\sqrt{g^2+f^2-c},$$

i.e. $\quad (gl+fm-n)^2 = (l^2+m^2)(g^2+f^2-c),$

or $\quad (l^2+m^2)c+n^2-(fl-gm)^2-2fmn-2gnl=0.$

Note. The above condition of tangency can also be obtained by the first method of (A).

6·9. Point of contact of a tangent line.

(A) *To find the point of contact when* $y=mx+c$ ... (1)
touches the ~~line~~ circle $x^2+y^2=a^2$. ... ... (2)

Let (x_1, y_1) be the point of contact. The equation of the tangent to (2) at (x_1, y_1) is

$$xx_1+yy_1=a^2. \quad \ldots \quad \ldots \quad (3)$$

Write (1) as $-mx+y=c$. ... ... (4)

Then (3) and (4) must be identical.

∴ Comparing coefficients of (3), (4),

$$\frac{x_1}{-m}=\frac{y_1}{1}=\frac{a^2}{c}.$$

$$\therefore \qquad x_1 = -\frac{a^2 m}{c},\ y_1 = \frac{a^2}{c}.$$

When (1) touches (2) since $c = a\sqrt{1+m^2}$,

the co-ordinates of point of contact can also be written as

$$\left(-\frac{ma}{\sqrt{1+m^2}},\ \frac{a}{\sqrt{1+m^2}}\right).$$

(B) *To find the point of contact when* $lx + my + n = 0$ (5)

touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0.$ (6)

Let (x_1, y_1) be the point of contact. The equation of the tangent to (6) at (x_1, y_1) is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0,$$

$$\text{or} \quad x(x_1 + g) + y(y_1 + f) + gx_1 + fy_1 + c = 0. \quad \dots \quad (7)$$

Hence (7) and (5) must be identical.

$$\therefore \quad \frac{x_1 + g}{l} = \frac{y_1 + f}{m} = \frac{gx_1 + fy_1 + c}{n}$$

$$= \frac{g(x_1 + g) + f(y_1 + f) - (gx_1 + fy_1 + c)}{gl + fm - n}$$

$$= \frac{g^2 + f^2 - c}{gl + fm - n}.$$

$$\therefore \qquad x_1 = \frac{l(g^2 + f^2 - c)}{gl + fm - n} - g,\ y_1 = \frac{m(g^2 + f^2 - c)}{gl + fm - n} - f.$$

6·10. Normal at a point.

To find the equation of the normal at (x_1, y_1) *to the circle*

(i) $x^2 + y^2 = a^2$;

(ii) $x^2 + y^2 + 2gx + 2fy + c = 0.$

The normal at any point P of a circle is the line which passes through P and is perp. to the tangent at P.

(i) Here the tangent at (x_1, y_1) is

$$xx_1 + yy_1 = a^2,$$

$$\text{i.e.} \qquad y = -\frac{x_1}{y_1}x + \frac{a^2}{y_1}. \qquad \dots \quad (1)$$

Any line through (x_1, y_1) is

$$y - y_1 = m(x - x_1). \quad \cdots \quad (2)$$

If (2) be normal at (x_1, y_1) it must be perp. to (1).

$$\therefore \quad m \times \left(-\frac{x_1}{y_1}\right) = -1, \quad i.e. \quad m = \frac{y_1}{x_1}$$

The equation of the normal at (x_1, y_1) is therefore

$$y - y_1 = \frac{y_1}{x_1}(x - x_1),$$

$$i.e. \quad y_1 x - x_1 y = 0. \quad \cdots \quad \cdots \quad (3)$$

(ii) Here the tangent at (x_1, y_1) is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0,$$

$$\text{or} \quad x(x_1 + g) + y(y_1 + f) + gx_1 + fy_1 + c = 0,$$

$$i.e. \quad y = -\frac{x_1 + g}{y_1 + f}x - \frac{gx_1 + fy_1 + c}{y_1 + f}. \quad \cdots \quad (4)$$

Any line through (x_1, y_1) is

$$y - y_1 = m(x - x_1). \quad \cdots \quad (5)$$

If (5) be normal at (x_1, y_1) it must be perp. to (4).

$$\therefore \quad m \times \left(-\frac{x_1 + g}{y_1 + f}\right) = -1, \quad i.e. \quad m = \frac{y_1 + f}{x_1 + g}.$$

The equation of the normal at (x_1, y_1) is therefore

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1),$$

$$i.e. \quad (y_1 + f)x - (x_1 + g)y - fx_1 + gy_1 = 0. \quad \cdots \quad (6)$$

Note. Since the equation (3) passes through (0, 0) and (6) passes through $(-g, -f)$, the centres of the respective circles, it follows that *the normal at any point of a circle passes through its centre.*

6·11. Number of tangents from a point.

To show that from any point there can be drawn two tangents to a circle.

Let (x_1, y_1) be the point and let the circle be

$$x^2 + y^2 = a^2. \quad \cdots \quad \cdots \quad (1)$$

The equation of any tangent to (1) is

$$y = mx + a\sqrt{1+m^2}, \quad \cdots \quad (2)$$

m being a variable parameter.

The problem of drawing a tangent from (x_1, y_1) to (1) reduces to that of choosing the parameter m so that (2) may pass through (x_1, y_1). The requisite relation to be fulfilled by m is accordingly

$$\therefore \quad y_1 = mx_1 + a\sqrt{1+m^2} \quad \cdots \quad (3)$$

or $\quad (y_1 - mx_1)^2 = a^2(1+m^2),$

or $\quad y_1{}^2 - 2mx_1y_1 + m^2x_1{}^2 = a^2(1+m^2),$

or $\quad m^2(x_1{}^2 - a^2) - 2mx_1y_1 + y_1{}^2 - a^2 = 0. \cdots (4)$

This equation being a quadratic in m gives two values of m, corresponding to each of which we have by substitution in (2) a tangent through (x_1, y_1).

Evidently *the two tangents from* (x_1, y_1) *will be real, coincident or imaginary* according as the two roots of (4) are real, coincident or imaginary,

i.e. according as $(2x_1y_1)^2 - 4(x_1{}^2 - a^2)(y_1{}^2 - a^2) \gtreqless 0,$

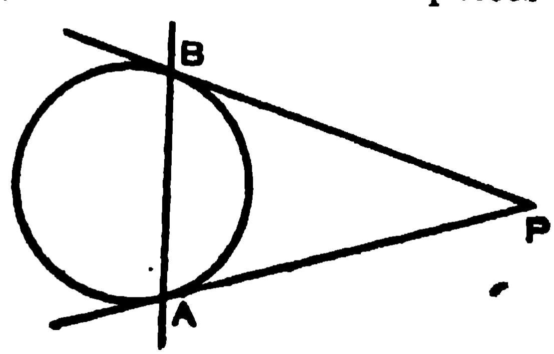
" $\quad a^2(x_1{}^2 + y_1{}^2 - a^2) \gtreqless 0,$

" $\quad x_1{}^2 + y_1{}^2 - a^2 \gtreqless 0,$

i.e. according as the point (x_1, y_1) *is outside, on, or inside the circle.* [*Art. 6·5*]

6·12. Pole and polar.

Def. The *polar* of a given point is the line which passes through the (real or imaginary) points of contact of tangents drawn from the given point; also the *pole* of any line is the point of intersection of tangents at the points (real or imaginary) in which the line meets the circle.



(A) *To find the polar of the point* (x_1, y_1) *w. r. t. the circle* :

(i) $x^2+y^2=a^2$,

(ii) $x^2+y^2+2gx+2fy+c=0$.

If the points of contact be $A(x_2, y_2)$, $B(x_3, y_3)$, the tangents at these points are

$$xx_2+yy_2=a^2,$$
$$xx_3+yy_3=a^2.$$

$\because$ both these tangents pass through $P(x_1, y_1)$, we have

$$x_1x_2+y_1y_2=a^2,$$
$$x_1x_3+y_1y_3=a^2.$$

These relations show that the two points $A(x_2, y_2)$, $B(x_3, y_3)$ are situated on the line

$$\mathbf{xx_1+yy_1=a^2},$$

which is therefore the required equation of the polar of the point (x_1, y_1).

(ii) If the points of contact be $A(x_2, y_2)$, $B(x_3, y_3)$ the tangents at these points are

$$xx_2+yy_2+g(x+x_2)+f(y+y_2)+c=0,$$
and $$xx_3+yy_3+g(x+x_3)+f(y+y_3)+c=0.$$

$\because$ both these tangents pass through $P(x_1, y_1)$ we have

$$x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0,$$
$$x_1x_3+y_1y_3+g(x_1+x_3)+f(y_1+y_3)+c=0.$$

These relations show that the two points $A(x_2, y_2)$ and $B(x_3, y_3)$ are situated on the line

$$\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0},$$

which is therefore the required equation of the polar of the point (x_1, y_1).

Note. It should be noted that when the point P lies within the circle the two tangents that can be drawn from P together with their points of contact A, B are *imaginary*, but the polar which is no other

than the line joining the *imaginary* points of contact is a *real* one. When P lies on the circumference of the circle, its polar coincides with the tangent at P.

(B) *To find the pole of the line $lx+my+n=0$ w. r. t. the circle*

(i) $x^2+y^2=a^2$.

(ii) $x^2+y^2+2gx+2fy+c=0$.

(i) Let (x_1, y_1) be the required pole of the line.

Now the polar of (x_1, y_1) w. r. t. (i) is

$$xx_1+yy_1-a^2=0.$$

$\therefore$ this must be identical with

$$lx+my+n=0.$$

$$\therefore \quad \frac{x_1}{l}=\frac{y_1}{m}=-\frac{a^2}{n},$$

whence $\quad x_1=-\dfrac{l}{n}a^2,\ y_1=-\dfrac{m}{n}a^2.$

(ii) Let (x_1, y_1) be the required pole of the line.

Then the polar of (x_1, y_1) w. r. t. (ii) is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0,$$

or $$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0.$$

This must therefore be identical with

$$lx+my+n=0.$$

$$\therefore \quad \frac{x_1+g}{l}=\frac{y_1+f}{m}=\frac{gx_1+fy_1+c}{n}$$

$$=\frac{g(x_1+g)+f(y_1+f)-(gx_1+fy_1+c)}{gl+fm-n}$$

$$=\frac{g^2+f^2-c}{gl+fm-n}.$$

$$\therefore \quad x_1=\frac{l(g^2+f^2-c)}{gl+fm-n}-g,\ \ y_1=\frac{m(g^2+f^2-c)}{gl+fm-n}-f.$$

6·13. Properties of pole and polar.

(i) *If the polar of a point P w. r. t. a circle passes through Q, the polar of Q passes through P.*

Let the co-ordinates of P, Q be (x_1, y_1), (x_2, y_2) and let the circle be $x^2 + y = a^2$. ... ... (1)

The polar of P with respect to (1) is $xx_1 + yy_1 = a^2 \cdots (2)$

The polar of Q with respect to (2) is $xx_2 + yy_2 = a^2 \cdots (3)$

If (2) passes through Q, then

$$x_2x_1 + y_2y_1 = a^2$$

which is exactly the condition that (3) passes through P.

(ii) *The point of intersection of any two lines is the pole of the line joining the poles of the lines.*

Let O be the intersection of the lines AB, CD and let P, Q be their poles. Since the polar of P *viz.* AB passes through O, the polar of O passes through P. Similarly the polar of O passes through Q. Hence PQ is the polar of O; in other words, O is the pole of the line PQ.

(iii) *If the pole of the line $lx + my + n = 0$ with respect to a circle lies on $l'x + m'y + n' = 0$, then the pole of $l'x + m'y + n' = 0$ lies on $lx + my + n = 0$.*

Let the circle be $x^2 + y^2 = a^2$.

Pole of $lx + ny + n = 0$ is $\left(-\frac{l}{n}a^2, -\frac{m}{n}a^2\right)$ ... (1)

Pole of $l'x + m'y + n' = 0$ is $\left(-\frac{l'}{n'}a^2, -\frac{m'}{n'}a^2\right)$ ... (2)

Since pole (1) lies on $l'x + m'y + n' = 0$.

$$\therefore \quad l'\left(-\frac{l}{n}a^2\right) + m'.\left(-\frac{m}{n}a^2\right) + n' = 0,$$

i.e. $$ll'a^2 + mm'a^2 - nn' = 0$$

which is exactly the condition that the pole (2) lies on

$$lx + my + n = 0.$$

6·14. Chord of contact.

From the definition of the polar of a point, it is clear that the polar of a point is nothing but the chord of contact *produced infinitely both ways.* Hence the *equation of the chord of contact of the tangents drawn from a point* (x_1, y_1) is to be obtained exactly in the same way as in the case of the polar of the point, and their respective equations for the circles (i) $x^2+y^2=a^2$,

$$\text{(ii) } x^2+y^2+2gx+2fy+c=0,$$

being, as in the case of polar

$$\mathbf{xx_1+yy_1=a^2},$$
$$\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0}.$$

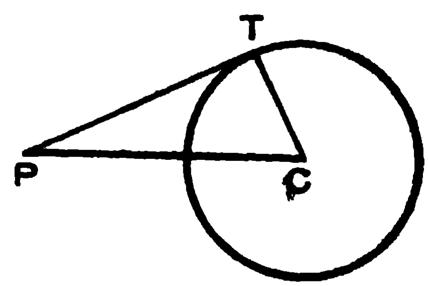
Note. It should be noted that the *equation of the tangent* at (x_1, y_1), *equation of the polar* of the point (x_1, y_1) and the *equation of the chord of the contact* of tangents drawn from (x_1, y_1) are all identical in form.

6·15. Length of the tangent.

To find the length of the tangent from an external point (x_1, y_1) *to the circle*

$$\text{(i) } x^2+y^2=a^2,$$
$$\text{(ii) } x^2+y^2+2gx+2fy+c=0.$$

Let $P(x_1, y_1)$ be the external point, PT the tangent and C the centre of the circle.



Then since $\angle PTC =$ a rt. $\angle$.

$\therefore \quad PT^2=PC^2-CT^2.$

(i) Here C is $(0, 0)$.

$\therefore \quad PC^2=x_1^2+y_1^2 \ ; \ CT^2=a^2.$

$\therefore \quad PT^2=\mathbf{x_1^2+y_1^2-a^2}.$

(ii) Here C is $(-g, -f)$;

$\therefore \quad PC^2=(x_1+g)^2+(y_1+f)^2 \ ; \ CT^2=g^2+f^2-c.$

$\therefore \quad PT^2=(x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$

$\qquad =\mathbf{x_1^2+y_1^2+2gx_1+2fy_1+c}.$

Thus, we see that *if* (x_1, y_1) *are substituted for* x, y *in the left-hand side of the equation of a circle which is written with the coefficients of* x^2 *and* y^2 *each equal to unity and with the right-hand member zero, we get the square of the length of the tangent from* (x_1, y_1).

Cor. The square of the tangent from the origin to the circle $x^2+y^2+2gx+2fy+c=0$ is c, which can, therefore, be taken as the *geometrical interpretation of* c.

6·16. Illustrative Examples.

Ex. 1. *Find the equation of the circle passing through three points* (x_1, y_1), (x_2, y_2), (x_3, y_3). [C. U. 1934]

Let $x^2+y^2+2gx+2fy+c=0\cdots(1)$ be the equation of the reqd. circle.

Since it passes through the 3 points,

$$\therefore \quad x_1^2+y_1^2+2gx_1+2fy_1+c=0. \quad \cdots \quad \cdots \quad (2)$$

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0. \quad \cdots \quad \cdots \quad (3)$$

$$x_3^2+y_3^2+2gx_3+2fy_3+c=0. \quad \cdots \quad \cdots \quad (4)$$

Eliminating the unknown quantities g, f, c between (1), (2), (3), (4), we have the reqd. equation of the circle

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Note. In numerical examples, it would be convenient to obtain the actual values of g, f, c from (2), (3), (4) and substitute them in (1), as shown below.

Ex. 2. *Find the equation of the circle which passes through the points* $(3, 4)$, $(3, -6)$, $(-1, 2)$.

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Since it passes through the 3 points, we have

$$25+6g+8f+c=0.$$

$$45+6g-12f+c=0.$$

$$5-2g+4f+c=0.$$

From the above 3 equations we easily obtain $g=-3$, $f=1$, $c=-15$. Hence the reqd. circle is

$$x^2+y^2-6x+2y-15=0.$$

Ex. 3. *Find the equation of the circle described on the line joining the points (x_1, y_1), (x_2, y_2) as diameter.*

Let $P(h, k)$ be any point on the circle and $A(x_1, y_1)$, $B(x_2, y_2)$ be the given points. The equations of PA, PB are

$$y-k=\frac{k-y_1}{h-x_1}(x-h),$$

and $$y-k=\frac{k-y_2}{h-x_2}(x-h).$$

Since APB is a semicircle, PA, PB are at right angles and hence the product of their m's $=-1$.

$$\therefore \quad \frac{k-y_1}{h-x_1}\cdot\frac{k-y_2}{h-x_2}=-1,$$

or $$(h-x_1)(h-x_2)+(k-y_1)(k-y_2)=0.$$

This shows that (h, k), lies on the circle

$$(\mathbf{x}-\mathbf{x}_1)(\mathbf{x}-\mathbf{x}_2)+(\mathbf{y}-\mathbf{y}_1)(\mathbf{y}-\mathbf{y}_2)=\mathbf{0},$$

which is therefore the reqd. equation.

Ex. 4. *Find the length of the chord by the circle $x^2+y^2=a^2$ on the line $y=mx+c$.*

Let O be the centre of the circle and A, B be the extremities of the chord. Draw OM perp. to AB.

Then $AB=2AM$; $AM^2=OA^2-OM^2$.

$$\therefore \quad AB^2=4AM^2=4\left\{a^2-\frac{c^2}{1+m^2}\right\}=4\left\{\frac{a^2(1+m^2)-c^2}{1+m^2}\right\},$$

whence AB can be obtained.

Examples VI(A)

1. Prove that the centres of the three circles $x^2+y^2=1$, $x^2+y^2+6x-2y=1$, $x^2+y^2-12x+4y=1$ are collinear.

2. Find the area of the triangle formed by joining the centres of the circles

$$x^2+y^2+2x+4y-4=0,\ x^2+y^2+6x+4y+9=0,$$
$$x^2+y^2-4x-4y+2=0.$$

3. If A, B, C be the centres of the three circles $x^2+y^2=27$, $x^2+y^2-10y-12=0$, $x^2+y^2-5\sqrt{3}x-5y-35=0$, verify that the triangle ABC is equilateral.

4. If A, B, C, D be the centres of the four circles, (i) $x^2+y^2-24x-10y+25=0$, (ii) $x^2+y^2+5x-12y-37=0$, (iii) $x^2+y^2+24x+10y+144=0$, (iv) $x^2+y^2-5x+12y-20=0$, show that the quadrilateral is a rhombus whose diagonals cross each other at the origin.

[*Show that AC, BD bisect each other orthogonally at the origin.*]

5. Show that the two circles

(i) $x^2+y^2-8x+10y+5=0$, $x^2+y^2+2x-14y+1=0$ touch each other externally.

(ii) $x^2+y^2-14x+2y-71=0$, $x^2+y^2-8x+10y+5=0$ touch each other internally.

6. Find the equations of the circumcircles of the triangle whose vertices are

(i) (2, 1), (1, 2), (8, 9) ;
(ii) (3, 0), (12, 0), (0, 6).

7. Find the equation of the nine-point circle of the triangle whose vertices are (2, 4), (4, 6), (6, 6).

[*Nine-point circle passes through the mid-points of the sides of a triangle.*]

8. Find the equation of the circle which is concentric with the circle $x^2+y^2-8x+12y+15=0$ and passes through (5, 4).

9. (i) Find the position of the point $(2, -3)$ *w. r. t.* the circle $x^2+y^2+2x-4y-9=0$.

(ii) Does the line $3x+4y+12=0$ intersect the circle $x^2+y^2-4x-6y-12=0$?

10. Does the equation $x^2+y^2-2x+2y+8=0$ represent a *real* circle ?

11. Show that the equaition of the circle through the origin and through the points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x^2+y^2 & x & y \\ x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \end{vmatrix}=0.$$

12. Find the intercepts made on the co-ordinate axes by the circle $x^2+y^2+2gx+2fy+c=0$. [$g^2 > c, f^2 > c$]

13. Show that the line $3x+4y+7=0$ touches the circle $x^2+y^2-4x-6y-12=0$ and find its point of contact. [C.U.]

14. Find the equation of the tangent to the circle $(x-h)^2+(y-k)^2=a^2$ at the point $(h+a\cos\alpha, k+a\sin\alpha)$.

15. If the line $x\cos\alpha+y\sin\alpha=p$ touches the circle $(x-a)^2+(y-b)^2=r^2$ then $a\cos\alpha+b\sin\alpha=p\pm r$.

16. Show that the condition for the tangency of the line $x\cos\alpha+y\sin\alpha-p=0$ to the circle $x^2+y^2+2gx+2fy+c=0$ is $p^2+2p(g\cos\alpha+f\sin\alpha)-(g\sin\alpha-f\cos\alpha)^2+c=0$.

17. If the tangents at (x_1, y_1) and (x_2, y_2) on the circle $x^2+y^2+2gx+2fy+c=0$ are perp., show that

$$x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+g^2+f^2=0.$$

18. Show that the circle $x^2+y^2-2ax-2ay+a^2=0$ touches both the co-ordinate axes and find their points of contact.

19. Find the equations of the tangents to the circle $x^2+y^2-2x-4y-4=0$, which are (i) perpendicular, (ii) parallel to the line $3x-4y-1=0$.

20. Find the points on the circle $x^2+y^2-2x+6y-58=0$ at which the tangents are parallel to the line $x+4y-7=0$.

21. Find the equation of the pair of lines joining the origin to the points of intersection of the line $\frac{x}{a}+\frac{y}{b}=1$

with the circle $x^2+y^2=c^2$ and hence deduce that if the line is a tangent to the circle, then $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$.

[*Use the condition the pair of lines become coincident.*]

22. (i) Find the length of the chord intercepted by the circle $x^2+y^2-12x+16y-69=0$ on the line $3x-4y+10=0$.

(ii) Show that the length of the least chord of the circle $x^2+y^2+2gx+2fy+c=0$ which passes through an internal point (x_1, y_1) is

$$2\sqrt{\{-(x_1{}^2+y_1{}^2+2gx_1+2fy_1+c)\}}.$$

[*See Ex. 4, Art. 6·16.*]

23. Prove that the equations

$$\{(y-\beta)-m(x-\alpha)\}^2=a^2(1+m^2),$$

and $$\{m(y-\beta)+(x-\alpha)\}^2=a^2(1+m^2)$$

represent the sides of a square circumscribed about the circle

$$(x-\alpha)^2+(y-\beta)^2=a^2.$$

[*The two equations break up into 4 lines each of which is a tangent to the circle.*]

24. Find the condition that $lx+my+n=0$, should be a normal to the circle $x^2+y^2+2gx+2fy+c=0$.

✓**25.** (i) Find the pole of the line $2x-y+10=0$ *w. r. t.* the circle $x^2+y^2-7x+5y-1=0$.

(ii) Show that the polar of the point (h, k) *w. r. t.* the circle $x^2+y^2-2\lambda x+c=0$ where λ is a variable parameter, always passes through a fixed point.

✓**26.** Prove analytically that the polar of a point *w. r. t.* a circle is perp. to the line joining the point to the centre.

27. Verify that the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) will be *collinear* if and only if their polars *w.r.t.* the circle $x^2+y^2=a^2$ are *concurrent.*

28. Show that the distances of two points each from the polar of the other *w.r.t.* a circle are to one another as the distances of the points from the centre of the circle.

29. Find the length of the tangent from the point (7, 2) to the circle $2x^2+2y^2+5x+y-15=0$.

30. Find the locus of a point the tangents from which to two circles are (i) equal, (ii) in a constant ratio $m:n$.

31. Show that the lengths of the tangents from any point on the line $3x-8y+2=0$ to the circles $x^2+y^2-4x+6y+8=0$, and $x^2+y^2+2x-10y+12=0$ are equal.

32. Show that the length of the tangent drawn from any point on the circle $x^2+y^2+2gx+2fy+c=0$ to the circle $x^2+y^2+2gx+2fy+c'=0$ is $\sqrt{c'-c}$.

33. Find the equation of the circle passing thorugh the common points of $x^2+y^2-2x-4y-4=0$, $2x+3y+1=0$ and the origin.

34. Find the equation of the circle which passes through the common points of the circle $S\equiv x^2+y^2+2gx+2fy+c=0$ and the line $L\equiv lx+my-1=0$ and the origin.

[*$S+\lambda L=0$ is the equation of any circle through the common points.*]

35. Find the equation of the circle which has for its diameter the chord cut off on the line $L\equiv lx+my-1=0$ by the circle $S\equiv x^2+y^2-a^2=0$.

[*Express the condition that the circle $S+\lambda L=0$ has its centre on $lx+my=1$.*]

36. Show that a circle can be inscribed in the quadrilateral whose sides are

$$x=0,\ y=0,\ x\cos\alpha+y\sin\alpha=p,\ x\cos\alpha'+y\sin\alpha'=p',$$

if
$$p(1+\sin\alpha'+\cos\alpha')=p'(1+\sin\alpha+\cos\alpha).$$

37. Show that the *necessary* and *sufficient* condition for four given poins (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) to be concylic is

$$\begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{vmatrix}=0.$$

6·17. Equation of a chord in terms of its middle point.

To find the equation of the chord of the circle $x^2+y^2=a^2$ ⋯ (1) *whose middle point is* (x_1, y_1).

The equation of any chord through (x_1, y_1) is

$$y-y_1=m(x-x_1). \quad \cdots \quad (7)$$

The line joining the centre of the circle and (x_1, y_1)

is $$y=\frac{y_1}{x_1}x. \quad \cdots \quad \cdots \quad (8)$$

Since (7) and (8) are perp.

$$\therefore \quad m\cdot\frac{y_1}{x_1}=-1, \text{ i.e. } m=-\frac{x_1}{y_1}.$$

∴ the required equation of the chord is

$$y-y_1=-\frac{x_1}{y_1}(x-x_1),$$

i.e. $\quad \mathbf{(x-x_1)x_1+(y-y_1)y_1=0.}$

Similarly the equation of the chord of the circle $x^2+y^2+2gx+2fy+c=0$ whose middle point is (x_1, y_1) is

$$(x-x_1)(x_1+g)+(y-y_1)(y_1+f)=0.$$

6·18. Pair of tangents from a point.

To find the equation of the pair of tangents from a point (x_1, y_1) *to the circle* $x^2+y^2=a^2$.

Let the equation of any line through (x_1, y_1) be

$$y-y_1=m(x-x_1). \quad \cdots \quad (1)$$

If it be a tangent to $x^2+y^2=a^2$

$$\therefore \quad \frac{(mx_1-y_1)^2}{1+m^2}=a^2,$$

or $$(mx_1-y_1)^2=a^2(1+m^2). \quad \cdots \quad (2)$$

Now eliminating m between (1) and (2), we get the required equation of tangents to be

$$\{x_1(y-y_1)-y_1(x-x_1)\}^2=a^2\{(x-x_1)^2+(y-y_1)^2\},$$

or $\quad (x_1y-y_1x)^2=a^2\{(x-x_1)^2+(y-y_1)^2\}.$

This when simplified can be put in the standard form

$$(x^2+y^2-a^2)(x_1{}^2+y_1{}^2-a^2)=(xx_1+yy_1-a^2)^2.$$

6·19. Parametric representation.

Since $x=a\cos\theta$, $y=a\sin\theta$, satisfy the equation of the circle $x^2+y^2=a^2$, whatever be the value of θ, co-ordinates of any point on the above circle can be represented by

$$\mathbf{x}=\mathbf{a}\cos\theta$$

$$\mathbf{y}=\mathbf{a}\sin\theta$$

which are called *parametric* equations, θ being called a *parameter*.

By eliminating θ, we get the equation in x, y. The point whose co-ordinates are $a\cos\theta$, $a\sin\theta$, is for sake of brevity called the **point θ.**

Since $\dfrac{y}{x}=\tan\theta$, or $y=x\tan\theta$, θ is the angle which the line joining the centre to the point of contact makes with the x-axis. This is sometimes spoken of as the *geometrical interpretation* of the parameter θ.

6·20. Equation of the chord in parametric co-ordinates.

To find the equation of the chord of the circle $x^2+y^2=a^2$ joining the points $(a\cos\theta, a\sin\theta)$ and $(a\cos\varphi, a\sin\varphi)$.

The equation of the line joining the points

$$\frac{x-a\cos\theta}{a(\cos\varphi-\cos\theta)}=\frac{y-a\sin\theta}{a(\sin\varphi-\sin\theta)},$$

or $$\frac{x - a\cos\theta}{2\sin\frac{1}{2}(\theta+\varphi)\sin\frac{1}{2}(\theta-\varphi)}$$

$$= \frac{y - a\sin\theta}{-2\cos\frac{1}{2}(\theta+\varphi)\sin\frac{1}{2}(\theta-\varphi)},$$

or $$\frac{x - a\cos\theta}{\sin\frac{1}{2}(\theta+\varphi)} = -\frac{y - a\sin\theta}{\cos\frac{1}{2}(\theta+\varphi)},$$

or $$x\cos\tfrac{1}{2}(\theta+\varphi) + y\sin\tfrac{1}{2}(\theta+\varphi)$$

$$= \{a\cos\theta\cos\tfrac{1}{2}(\theta+\varphi) + \sin\theta\sin\tfrac{1}{2}(\theta+\varphi)\}$$

$$= a\cos\{\theta - \tfrac{1}{2}(\theta+\varphi)\}$$

$$= a\cos\tfrac{1}{2}(\theta-\varphi).$$

Thus, the required equation of the chord is

$$x\cos\tfrac{1}{2}(\theta+\varphi) + y\sin\tfrac{1}{2}(\theta+\varphi) = a\cos\tfrac{1}{2}(\theta-\varphi).$$

Cor. 1. Putting $\varphi = \theta$ in the above equation of the chord *the equation of the tangent at* $(a\cos\theta, a\sin\theta)$ is

$$\mathbf{x\cos\theta + y\sin\theta = a.}$$

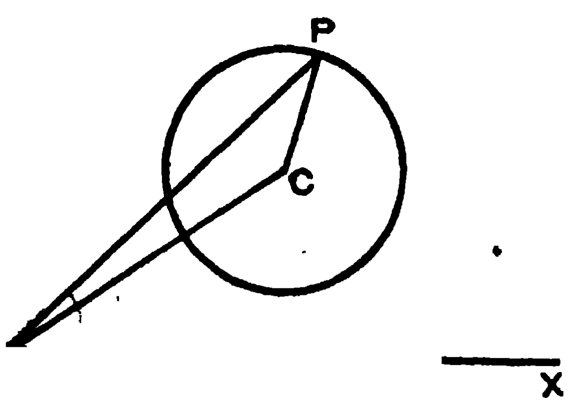
Thus, the line $x\cos\theta + y\sin\theta = a$ is a tangent to the circle $x^2 + y^2 = a^2$ for all values of θ.

Cor. 2. It can be easily shown directly or by change of origin that $(x-\alpha)\cos\theta + (y-\beta)\sin\theta = a$ is a tangent to the circle $(x-\alpha)^2 + (y-\beta)^2 = a^2$.

Note. The equation of the tangent at 'θ' can be obtained independently as shown in Art. 6·7 (*Alternative method*).

6·21. Polar equation of a circle.

To find the polar equation of a circle whose centre is (k, α) *and radius is* a.



Let (r, θ) be the polar co-ordinates of any point P on the circle. Then $\angle XOP = \theta$, $OP = r$.

Since,

$$\angle XOC = \alpha,\ OC = k.$$

$$\therefore\ \angle POC = \theta - \alpha.$$

Now, from $\triangle OPC$,

$$PC^2 = OP^2 + OC^2 - 2OP.OC \cos POC.$$

$$\therefore \quad a^2 = r^2 + k^2 - 2rk \cos(\theta - \alpha).$$

$$\therefore \quad r^2 - 2rk \cos(\theta - \alpha) + k^2 - a^2 = 0, \quad \ldots \quad (1)$$

or $$r^2 - 2rk \cos(\theta - \alpha) + c = 0, \quad \ldots \quad (2)$$

putting $c = k^2 - a^2$.

This being the relation between the polar co-ordinates of any point on the circle is the *polar equation* of the circle.

Cor. Any equation of the form

$$r^2 + r(A \cos \theta + B \sin \theta) + c = 0$$

represents a circle.

Note. If r_1, r_2 be the roots of (1), we have $r_1 r_2 = c^2 - a^2 = \text{const.}$ for all values of θ. Hence, if through a point O, a secant OP_1P_2 is drawn to meet the circle in P_1, P_2, then $OP_1 . OP_2$ is constant.

Special cases.

(i) *When the pole is on the circumference.*

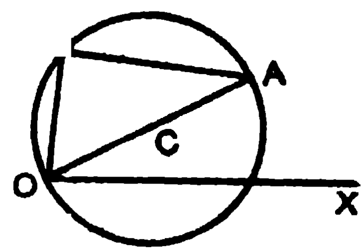
Now co-ordinates of C are (a, α).

From $\triangle OPA$,

$$OP = OA \cos POA,$$

i.e. $$r = 2a \cos(\theta - \alpha).$$

This is the required equation.



Cor. The above equation can be written as

$$r = A \cos \theta + B \sin \theta.$$

(ii) ***When the pole is on the circumference and the initial line passes through the centre.***

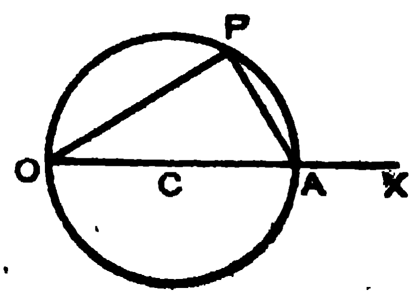
Now, co-ordinates of C are (a, o).

From $\triangle OPA$,

$$OP = OA \cos POA,$$

i.e. $\mathbf{r = 2a \cos \theta}$.

This is the required equation.



(iii) *When the centre of the circle is taken as the pole, whatever be the direction of the initial line.*

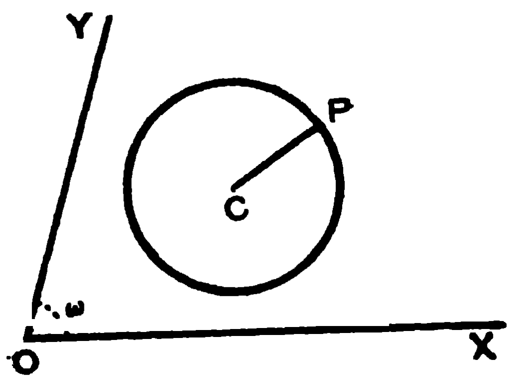
Now, $OP=a$.

$\therefore \quad r=a$.

This is the required equation.

6·22. Equation of a circle (*oblique axes*).

Let the co-ordinates of the centre and any point on the circle of radius a be (α, β) and (x, y) respectively *w. r. t.* a pair of oblique axes OX, OY, ω being the angle between the axes.



Now $CP^2=a^2$.

$\therefore$ by Art. 2·4 (B),

$$(x-\alpha)^2+(y-\beta)^2+2(x-\alpha)(y-\beta)\cos\omega=a^2.$$

This is the reqd. equation.

6·22. Illustrative Example.

Ex. *Find the equation of the tangents from the origin to the circle*

$$x^2+y^2-14x+2y+25=0.$$

Let $y=mx$ ... (1) be any line through the origin. The abscissæ of the points of intersection of the line and the circle are given by $(1+m^2)x^2+x(2m-14)+25=0$. Hence the line will be a tangent to the circle, if it has equal roots,

$$i.e. \text{ if } \quad (2m-14)^2=4\times25(1+m^2),$$

$$\text{or} \quad 12m^2+7m-12=0. \quad \cdots \quad \cdots \quad (3)$$

Now eliminating m between (1) and (2), we get the tangents from the origin *viz.*,

$$12y^2+7xy-12x^2=0,$$

$$\text{or} \quad 12x^2-7xy-12y^2=0. \cdots \quad \cdots \quad (3)$$

Otherwise,

The polar of the origin *w. r. t.* the circle is $7x-y=25$. Now if the

equation of the circle is made homogeneous by means of this equation, we shall get the equations of the pair of tangents from the origin *viz.*,

$$x^2+y^2+(2y-14x)\frac{(7x-y)}{25}+25\left(\frac{7x-y}{25}\right)^2=0,$$

which when simplified leads to equation (3).

Examples VI(B)

1. A point moves so that the sum of the squares of its distances from the angular points of a triangle is constant. Show that the locus is a circle whose centre is the centroid of the triangle.

2. A point moves so that the sum of the squares of its distances from n fixed points is constant. Prove that its locus is a circle.

3. Find the locus of a point which is such that the tangents from it to two concentric circles are inversely as their radii.

4. If the sum of the squares of the tangents drawn from a variable point P to n given circles is constant ; prove that the locus of P is a circle.

5. Whatever be the value of α, prove that the locus of the point of intersection of the st. lines $x \cos \alpha + y \sin \alpha = a$ and $x \sin \alpha - y \cos \alpha = b$ is a circle. [*C.U. 1946*]

6. Show that the locus of the feet of the perpendiculars from a fixed point on a circle upon its diameter is another circle and determine its centre and radius.

7. Find the locus of the foot of the perp. from the origin on any tangent to the circle $x^2+y^2+2gx+2fy+c=0$.

8. Find the locus of the foot of the perp. from the origin upon any chord of the circle $x^2+y^2+2gx+2fy+c=0$ which subtends a right angle at the origin.

9. Show that when m varies, the locus of the pole of the line $y=mx+\dfrac{a}{m}$ *w.r.t.* the circle $(x-a)^2+y^2=b^2$, is another circle.

10. (i) Find the middle point of the chord intercepted by the circle $x^2 + y^2 = a^2$ on the line $lx + my + n = 0$.

(ii) Find the equation to that chord of the circle which is bisected at the point $(-2, 3)$.

11. Find the locus of the middle points of the chords of the circle which pass through the point (h, k).

12. Find the locus of the middle points of the chords of the circle $x^2 + y^2 = a^2$ which subtend a right angle at the origin.

13. Show that the locus of the middle points of the parallel chords of a circle is a line passing through the centre of the circle.

14. Show that the locus of the point, the tangents from which to a given circle are at right angles is another concentric circle.

15. Find the pair of tangents from (1, 5) to the circle $x^2 + y^2 = 13$.

16. Find the equation of the tangents from the origin to the circle $x^2 + y^2 + 10x + 10y + 40 = 0$.

17. Show that the area of the triangle formed by the tangents to the circle $x^2 + y^2 = a^2$ from (x_1, y_1) and their chord of contact is

$$a \cdot \frac{(x_1^2 + y_1^2 - a^2)^{\frac{3}{2}}}{x_1^2 + y_1^2}.$$

18. The parametric co-ordinates of a point P are

$$x = a.\frac{1 - t^2}{1 + t^2}, \quad y = a.\frac{2t}{1 + t^2}.$$

Show that the locus of P is a circle of radius a.

19. Find the centre and radius of the following circles

(i) $r = 2a \cos \theta + 2b \sin \theta$;

(ii) $r = 5(\sqrt{3} \cos \theta + \sin \theta)$.

20. Find the polar co-ordinates of the points of intersection of the line $r \cos \theta = a$ and the circle $r = 2a \cos \theta$.

21. Show that if $b^2c^2 + 2ac = 1$, the line :

$$\frac{1}{r} = a \cos \theta + b \sin \theta$$

touches the circle $r = 2c \cos \theta$.

22. Show that the circles $r = a \cos (\theta - \alpha)$ and $r = b \cos (\theta - \beta)$ intersect at an angle of $\alpha - \beta$.

23. Show that the circles $r = a \cos (\theta - \alpha)$ and $r = b \sin (\theta - \alpha)$ cut orthogonally.

24. Find the polar equation of the circle described upon the join of the points (r_1, θ_1), (r_2, θ_2) as diameter.

25. Show that the diameter of the circle passing through the pole and the points (a, α), (b, β) is of length

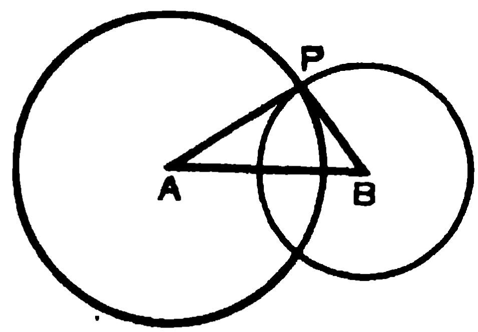
$$\frac{\sqrt{\{a^2 + b^2 - 2ab \cos (\alpha - \beta)\}}}{\sin (\alpha - \beta)}.$$

CHAPTER VII

SYSTEM OF CIRCLES

7·2. Condition for orthogonal intersection.

Def. Two circles are said to intersect orthogonally when the tangents at either of the two points of intersection are at right angles.



Suppose the two circles whose centres are A, B intersect orthogonally at P. Since the tangents to the two circles at P are at right angles, the radii AP, BP, being perp. to the respective tangents are also at right angles.

Hence $AP^2 + BP^2 = AB^2$.

Thus, *for two orthogonal circles the square of the distance between their centres is equal to the sum of the squares of their radii.*

Again, since PB is perp. to PA, PB is a tangent to the first circle. Thus, *for two orthogonal circles, the length of the tangent to the first circle from the centre of the second is equal to the radius of the second and vice versa.*

To find the condition that the two circles whose equations are

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$
$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

shall cut orthogonally.

First Method, (based on the first property)

When two circles cut orthogonally, the square of the distance between their centres is equal to the sum of the squares of their radii,

$$\therefore \quad (-g_1+g_2)^2+(-f_1+f_2)^2$$
$$=g_1^2+f_1^2-c_1+g_2^2+f_2^2-c_2,$$
$$i.e. \quad \mathbf{2g_1g_2+2f_1f_2=c_1+c_2}$$

as the condition for orthogonality of two circles.

The above condition is both *necessary* and *sufficient.*

Second Method.

From the second property, it follows that the two circles cut orthogonally if

$$g_2^2+f_2^2-2g_1g_2-2f_1f_2+c_1=g_2^2+f_2^2-c_2,$$
$$i.e. \text{ if } \quad 2g_1g_2+2f_1f_2=c_1+c_2.$$

Third Method, (based on definition)

Let (x', y') be the co-ordinates of a point of intersection of two circles. The equations of the tangents at this point are

$$xx'+yy'+g_1(x+x')+f_1(y+y')+c_1=0,$$
$$i.e. \quad (x'+g_1)x+(y'+f_1)y+g_1x'+f_1y'+c_1=0, \quad \dots \quad (1)$$
$$\text{and} \quad (x'+g_2)x+(y'+f_2)y+g_2x'+f_2y'+c_2=0. \quad \dots \quad (2)$$

Since (1) and (2) are at right angles, we have

$$(x'+g_1)(x'+g_2)+(y'+f_1)(y'+f_2)=0. \quad \dots \quad (3)$$

Since (x', y') is on both the circles,

$$x'^2+y'^2+2g_1x'+2f_1y'+c_1=0, \quad \dots \quad (4)$$
$$x'^2+y'^2+2g_2x'+2f_2y'+c_2=0. \quad \dots \quad (5)$$

Adding (4) and (5) and subtracting the result from (3) multiplied by 2 *i.e.*, thus eliminating x', y', we get the required condition of orthogonality.

7·2. Radical Axis.

Def. The radical axis of two circles is the locus of the point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

To find the equation of the radical axis of the circles

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0, \quad \cdots \quad (1)$$

$$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0. \quad \ldots \quad (2)$$

Let (x_1, y_1) be the co-ordinates of any point such that the tangents from it to the two circles (1) and (2) are equal.

$$\therefore \quad x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1$$
$$=x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2.$$

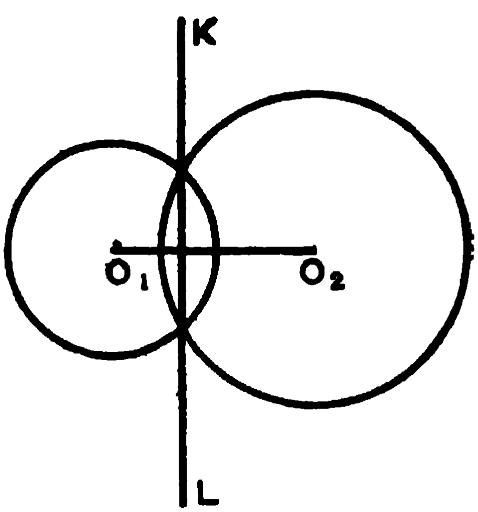
$$\therefore \quad 2(g_1-g_2)x_1+2(f_1-f_2)y_1+c_1-c_2=0.$$

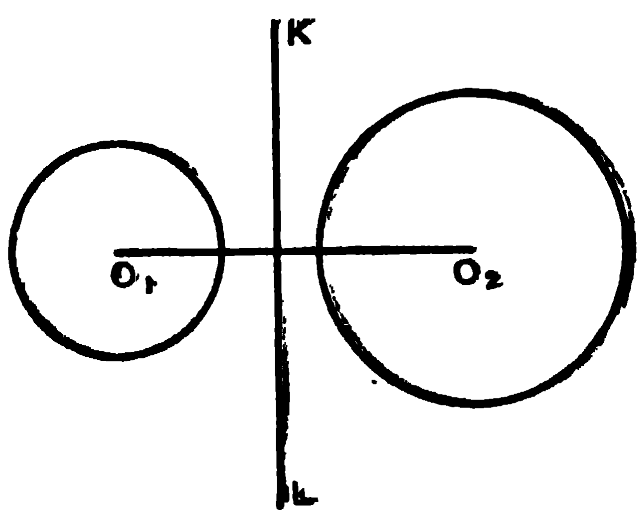
This shows that the locus of (x_1, y_1) is the line

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0 \quad \cdots \quad (3)$$

which is therefore the *equation of the radical axis*. Thus, *the equation of the radical axis is* $S_1-S_2=0$ (provided the coefficients of x^2 and y^2 are unity in both the equations).

Cor. 1. From the form of the equation of the radical axis *viz.* $S_1-S_2=0$, it appears that *the radical axis passes through the common points, (real or imaginary), of* $S_1=0$, $S_2=0$. Thus, the *radical axis is the common chord of the two circles.*





It should be carefully noted as shown in the above figures that the points of intersection may be *real* (as in Fig. 1) or imaginary (as in Fig. 2) but in both cases the radical axis a real line.

Cor. 2. It is also evident from above that *if two circles touch each other internally or externally the tangent at their point of contact is no other than their radical axis, and conversely* whenever the radical axis of two circles touches one of them, it must also touch the other and the circles themselves must touch each other.

7·3. Properties of Radical Axis.

(i) *The radical axis of two circles is perpendicular to the line of centres of the circles.*

Since the centres of the two circles are $(-g_1, -f_1)$ and $(-g_2, -f_2)$, the equation of the line joining them

is $$y+f_1=\frac{f_1-f_2}{g_1-g_2}(x+g_1). \quad \ldots \quad (4)$$

The 'm' of the line (3) is $-\dfrac{g_1-g_2}{f_1-f_2}$.

$\therefore$ the product of the m's of the lines (3) and (4) $= -1$.

Hence the result.

(ii) *The radical axes of three circles taken in pairs are concurrent.*

Let the equations of the three circles be

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0, \quad \ldots \quad (1)$$
$$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0, \quad \ldots \quad (2)$$
$$S_3 \equiv x^2+y^2+2g_3x+2f_3y+c_3=0. \quad \ldots \quad (3)$$

The radical axes of the circles taken in pairs are given by

$$S_1-S_2 \equiv 2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0,$$
$$S_2-S_3 \equiv 2(g_2-g_3)x+2(f_2-f_3)y+c_2-c_3=0,$$
$$S_3-S_1 \equiv 2(g_3-g_1)x+2(f_3-f_1)y+c_3-c_1=0.$$

Since on adding the above three equations, their sum vanishes identically, the three radical axes meet in a point. [*See Art. 3·10.*]

The point of concurrence of the radical axes of three circles taken in pairs is called the **radical centre** of the

three circles. Since the radical centre is given by $S_1 = S_2 = S_3$, this shows that the *lengths of the tangents from the radical centre to the three circles are equal.*

7·4. Centre-locus of a circle cutting two circles orthogonally.

Let the equations of two given circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0, \quad \cdots \quad (1)$$
$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0. \quad \cdots \quad (2)$$

Let the equation of the circle which cuts both of them orthogonally be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0. \quad \cdots \quad (3)$$

Since (3) cuts (1) and (2) orthogonally,

$$\therefore \quad 2gg_1 + 2ff_1 - c - c_1 = 0, \quad \cdots \quad (4)$$
$$2gg_2 + 2ff_2 - c - c_2 = 0. \quad \cdots \quad (5)$$

Subtracting (5) from (4), we get

$$2g(g_1 - g_2) + 2f(f_1 - f_2) - (c_1 - c_2) = 0,$$
$$\text{or} \quad (-g).2(g_1 - g_2) + (-f).2(f_1 - f_2) + c_1 - c_2 = 0.$$

This shows that $(-g, -f)$, the centre of the circle (3) lies on the line

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0,$$
$$\textit{i.e.} \quad \text{on the line} \qquad S_1 - S_2 = 0,$$

which is the radical axis of (1) and (2).

Thus, *the locus of the centre of the circle which cuts two given circles orthogonally is the radical axis of the two circles.*

Hence it follows automatically that the *radical centre of three given circles is designable as the centre of the uniquely determinate circle which cuts all of them orthogonally.*

Also it should be noted that *if a circle cuts three given circles orthogonally, its centre is the radical centre of the*

three given circles and its radius is equal to the length of the tangent drawn from the radical centre to any of the circles.
[See Art. 7·1]

Hence *the equation of the radical circle of three given circles* $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ may be written as

$$(x-\alpha)^2+(y-\beta)^2=r^2$$

where (α, β) is the radical centre of the given circles and r is the length of the tangent from (α, β) to any one of these circles.

This form of the equation is commonly used is solving numerical examples. For another method see Ex. 2, Illustrative Examples, Art. 7 8.

7·5. Circle through the intersection of two circles.

Let the equations of the given circles be

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0, \quad \ldots \quad (1)$$

$$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0, \quad \ldots \quad (2)$$

then $S_1 - \lambda S_2 = 0, \quad \ldots \quad \ldots \quad (3)$

(where λ is a variable parameter) represents any circle through the intersections of (1) and (2).

Co-ordinates of those points which satisfy simultaneously (1) and (2) *i.e.*, the co-ordinates of their points of intersection also satisfy (3). Hence (3) passes through the intersection of (1) and (2). Moreover (3) obviously is the equation of a circle, as will be seen from the expanded form of the equation *viz.*

$$x^2+y^2+2g_1x+\cdots-\lambda(x^2+y^2+2g_2x+\cdots)=0,$$

i.e. $(1-\lambda)(x^2+y^2)+2(g_1-\lambda g_2)x+2(f_1-\lambda f_2)y+c_1-\lambda c_2=0.$

By varying λ, we shall obtain the equations of the different circles, all passing through the two points of intersection of (1) and (2) and hence *having a common radical axis.*

Note. It should be noted that when $\lambda=1$, the circle *viz.* S_1-S_2 degenerates into a line *viz.*, the common chord (or radical axis) of the circles.

If $S=0$, and $L=0$ represent a circle and a line, it can be shown similarly that $S-\lambda L=0$ is the general equation of a circle passing through the common points of $S=0$, $L=0$.

7·6. Co-axial Circles.

Def. A system of circles every pair of which has the same radical axis are called **Co-axial circles.**

Thus a system of co-axial circles has a common radical axis and passes through two fixed points, real or imaginary.

The general equation of a system of co-axial circles can be written in either of the forms

(i) $S_1-\lambda S_2=0$, where $S_1=0$, $S_2=0$ are any two circles of the system, having $S_1-S_2=0$ as the radical axis.

(ii) $S-\lambda L=0$, where $S=0$ is any circle of the system and $L=0$ is the equation of the radical axis.

Since the radical axis of any pair of circles is perpendicular to their line of centres, it follows that *the centres of circles of a co-axial system all lie on a line perp. to the radical axis.*

Equation of a system of co-axial circles, (special choice of axes).

Let us take the line of centres as the x-axis and the radical axis as the y-axis.

The equation to any circle with its centre on the x-axis is of the form

$$(x-g)^2+y^2=r^2,$$

i.e., of the form $x^2+y^2-2gx+c=0$. ... (1)

Any point on the radical axes is $(0, y_1)$. The square of the tangent from it to the circle (1) is y_1^2+c. Since this is to be the same for *all* circles of the system, c must be the same for all circles and g different.

Hence the equation of a system of co-axial circles with the above choice of axes is of the form

$$x^2 + y^2 + 2\lambda x + c = 0,$$

where λ is variable but c is a fixed constant.

7·7. Limiting points.

It follows from Art. 7·5 that if C be the centre and R be the radius of the circle $S_1 - \lambda S_2 = 0$, ... (1)

then C is $\left(-\frac{g_1 - \lambda g_2}{1-\lambda}, -\frac{f_1 - \lambda f_2}{1-\lambda}\right)$ ... (2)

and $R^2 = \frac{(g_1 - \lambda g_2)^2 + (f_1 - \lambda f_2)^2 - (1-\lambda)(c_1 - \lambda c_2)}{(1-\lambda)^2}$. (3)

When $R = 0$, the numerator of the right side of (2) is zero which when simplified gives a quadratic in λ. Thus there are two values of λ for which the circle $S_1 - \lambda S_2 = 0$ reduces to a circle of zero radius *i.e.* a point. In other words, there are two circles of zero radius belonging to the co-axial system. These are called *limiting points* of the system.

If λ_1, λ_2 are the two roots of the quadratic in λ, the two limiting points would be obtained either by substituting their values in (2) or in (1) which would ultimately reduce (in either case) to the form $(x-h)^2 + (y-k)^2 = 0$, giving (h, k) as the limiting point.

Limiting points of $x^2 + y^2 + 2\lambda x + c = 0$.

The equation may be written as

$$(x+\lambda)^2 + y^2 = \lambda^2 - c,$$

when $\lambda^2 - c = 0$ or $\lambda = \pm\sqrt{c}$, the circle reduces to a point. Hence the point circles of the system are given by

$$(x - \sqrt{c})^2 + y^2 = 0 \text{ and } (x + \sqrt{c})^2 + y^2 = 0,$$

$\therefore$ the limiting points are $(\sqrt{c}, 0)$, $(-\sqrt{c}, 0)$.

7·8. Properties of Limiting points.

(1) *The polar of either limiting point w. r. t. any circle of the system passes through the other.*

The polar of $(\sqrt{c}, 0)$ *w. r. t.* $x^2+y^2+2\lambda x+c=0$ is

$$x\sqrt{c}+\lambda(x+\sqrt{c})+c=0, \text{ or } x+\sqrt{c}=0,$$

which obviously passes through $(-\sqrt{c}, 0)$.

Cor. The polar of a limiting point *w. r. t.* any circle of the co-axial system is the same for all circles of the system.

(2) *Every circle through the limiting points of a co-axial system is orthogonal to all circles of the system.*

The general equation of a circle, passing through the limiting points $(\sqrt{c}, 0)$, $(-\sqrt{c}, 0)$, is $x^2+y^2+2fy-c=0$, when f is a variable parameter. Now the circles

$$x^2+y^2+2fy-c=0,$$
$$x^2+y^2+2\lambda x+c=0,$$

can easily be shown to be orthogonal.

7·9. Common tangents.

We shall briefly indicate here the method of drawing common tangents to two circles

$$(x-\alpha)^2+(y-\beta)^2=r^2, \quad (x-\alpha')^2+(y-\beta')^2=r'^2.$$

Let $\qquad lx+my+n=0 \quad \dots \quad \dots \quad (1)$

be the equation of the required common tangent. Since it touches the first circle,

$$\frac{l\alpha+m\beta+n}{\sqrt{l^2+m^2}}=\pm r,$$

i.e. $\qquad (l\alpha+m\beta+n)^2=r^2(l^2+m^2). \quad \dots \quad (2)$

Similarly, since it touches the 2nd circle

$$(l\alpha'+m\beta'+n)^2=r'^2(l^2+m^2). \quad \dots \quad (3)$$

(2) and (3) being homogeneous quadratics in l, m, n, by solving them we shall get 4 sets of proportional values of l, m, n (real or imaginary) and substituting them in (1), we would get 4 common tangents, (real or imaginary) to the above two circles. The method is illustrated in Ex. 3, Art. 7·10.

It is geometrically obvious that a common tangent to two circles cuts the join of centres internally or externally in the ratio of the radii.

Otherwise :

The equation of any tangent to the first circle is

$$(x-\alpha)\cos\theta+(y-\beta)\sin\theta=r. \qquad \dots \quad (1)$$

[Art. 6·20, Cor. 2]

If it touches the second, then

$$(\alpha'-\alpha)\cos\theta+(\beta'-\beta)\sin\theta=r\pm r'. \qquad \dots \quad (2)$$

Eliminate θ between (1) and (2), and we shall get the joint equation of two pairs of common tangents. This method is illustrated in Ex. 3, Art. 7·10.

7·10. Illustrative Examples.

Ex. 1. *Show that the circles* $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x-6y+14=0$ *touch at the point* $(3, -1)$. [*C. U. 1932*]

The common chord of the circles is $(x^2+y^2-4x+6y+8)-(x^2+y^2-10x-6y+14)=0$ *i.e.*, $x+2y-1=0$.

The ordinates of the points of intersection of this line and the first circle are given by

$$(1-2y)^2+y^2-4(1-2y)+6y+8=0,$$

i.e., $\quad y^2+2y+1=0, \quad$ *i.e.*, $\quad (y+1)^2=0,$

i.e. $\quad y=-1, -1.$

$\therefore \quad x=1-2y=3, 3.$

Thus the common chord touches the circle (1) at $(3, -1)$; hence it also touches (2) at the same point.

$\therefore$ the two circles touch at $(3, -1)$.

Alternative Method.

That the two circles touch can be shown from the fact that distance between their centres is equal to the sum or difference of their radii and the point of contact can be obtained from the fact that it divides the line joining the centres in the ratio of their radii.

Ex. 2. *Find the equation of the circle which cuts orthogonally the three circles*

$$x^2+y^2+4x+7=0, \quad \cdots \quad \cdots \quad (1)$$
$$2x^2+2y^2+3x+5y+9=0, \quad \cdots \quad \cdots \quad (2)$$
$$x^2+y^2+y=0. \quad \cdots \quad \cdots \quad (3)$$

The radical axis of (1) and (2) is

$$(x^2+y^2+4x+7)-(x^2+y^2+\tfrac{3}{2}x+\tfrac{5}{2}y+\tfrac{9}{2})=0,$$

i.e. $$x-y+1=0. \quad \cdots \quad \cdots \quad (4)$$

The radical axis of (1) and (3) is

$$4x-y+7=0. \quad \cdots \quad \cdots \quad (5)$$

$\therefore$ The radical centre *i.e.* the point of intersection of (4) and (5) is $(-2, -1)$. The length of the tangent from $(-2, -1)$ to the circle (3) is 2.

$\therefore$ by Art. 7·4, the equation of the required circle is

$$(x+2)^2+(y+1)^2=2^2,$$

i.e. $$x^2+y^2+4x+2y+1=0.$$

Otherwise :

Let the equation of the circle which cuts the 3 circles orthogonally, be $$x^2+y^2+2gx+2fy+c=0.$$

Since it cuts orthogonally the circles (1), (2), (3).

$\therefore$ by Art. 7·1, $4g=c+7,$

$$\tfrac{3}{2}g+\tfrac{5}{2}f=c+\tfrac{9}{2},$$
$$f=c.$$

From which we easily get $g=2$, $f=1$, $c=1$. Hence follows the equation of the required circle.

Ex. 3. *Find the common tangents to the circles*

$$x^2+y^2=9, \text{ and } x^2+y^2-16x+2y+49=0,$$

The 2nd circle can be written as $(x-8)^2+(y+1)^2=4^2$.

Let $lx+my+n=0$, ... (1) be the tangent to both the circles.

Then, $$n^2=9(l^2+m^2), \quad \cdots \quad \cdots \quad (2)$$

and $$(8l-m+n)^2=16(l^2+m^2). \quad \cdots \quad \cdots \quad (3)$$

From (2) and (3), $(8l-m+n)^2=\frac{16}{9}n^2$.

$$\therefore \quad 8l-m+n=\pm\tfrac{4}{3}n.$$

Taking $+$**sign,** $\quad 8l-m=\frac{1}{3}n. \quad \cdots \quad \cdots \quad (5)$

Taking $-$**sign,** $\quad 8l-m=-\frac{7}{3}n. \quad \cdots \quad (6)$

Eliminating n **between (5) and (2), we get** $l=0$ **or** $l=\frac{16}{63}m$.

When $l=0$**, from (5), we get** $m=-\frac{1}{3}n$.

Again, when $l=\frac{16}{63}m$**,** *i.e.* $m=\frac{63}{16}l$**, from (5),** $n=\frac{195}{16}l$.

$\therefore$ **the equations of 2 common tangents are**

$$-\tfrac{1}{3}ny+n=0 \text{ or } y=3, \quad \cdots \quad \cdots \quad (1)$$

and $\quad lx+\frac{63}{16}ly+\frac{195}{16}l=0$, or $16x+63y+195=0. \quad \cdots \quad (2)$

Similarly combining (6) and (2) we shall get

$$m=\tfrac{5}{12}l,\ n=-\tfrac{13}{4}l \text{ and } m=-\tfrac{3}{4}l,\ n=-\tfrac{15}{4}l.$$

$\therefore$ **the equations of the other two common tangents are**

$$12x+5y-39=0, \quad \cdots \quad \cdots \quad (3)$$

and $$4x-3y-15=0. \quad \cdots \quad \cdots \quad (4)$$

***Otherwise*:**

Any tangent of the first circle is

$$x\cos\theta+y\sin\theta-3=0. \quad \cdots \quad \cdots \quad (1)$$

If it touches the 2nd circle

$$8\cos 6-\sin\theta-3=0\pm 4. \quad \cdots \quad (2)$$

Taking $-$ **sign,** $8\cos\theta-\sin\theta+1=0. \quad \cdots \quad \cdots \quad (3)$

Solving (1) and (3), for $\cos\theta$, $\sin\theta$ **by the rule of cross-multiplication and squaring and adding we get**

$$(y-3)^2+(x+24)^2=(x+8y)^2,$$

$$\text{or } 63y^2+16xy-48x+6y-585=0,$$

$$\text{or } (y-3)(63y+16x+195)=0.$$

Similarly, taking $+$ **sign, we get the other pair of tangents.**

Examples VII

1. If the circles $x^2+y^2+kx+3y-5=0$ and $x^2+y^2+5x+ky+7=0$ cut orthogonally, find k.

2. Prove that the circles $x^2+y^2-2px-2qy=2pq$ and $x^2+y^2+2qx+2py=2pq$ are orthogonal.

3. Find the equation of a circle which passes through the origin and cuts orthogonally the circles $x^2+y^2-8y+12=0$ and $x^2+y^2-4x-6y-3=0$. [*M.U.*]

4. Verify that the radical axis of the pair of circles $x^2+y^2+2x-2y-5=0$ and $x^2+y^2+7x-2y-13=0$ is perpendicular to that of the circles

$$4(x^2+y^2)-6x-y+5=0 \text{ and } 8(x^2+y^2)-12x+3y+7=0.$$

5. Find the radical axis of the circles $x^2+y^2+2ax+c^2=0$ and $x^2+y^2+2by+c^2=0$ and hence deduce that the condition that they touch other, is $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$.

[*Obtain the condition that their radical axis touches one of them.*]

6. (i) Find the radical centre of the three circles $x^2+y^2+x+2y+3=0$, $x^2+y^2+2x+4y+5=0$, $x^2+y^2-7x-8y-9=0$. [*C.U.*]

(ii) Find the radical centre of the three circles $x^2+y^2+4x+7=0$, $2(x^2+y^2)+3x+5y+9=0$ and $x^2+y^2+y=0$. Find the length of the tangent from the radical centre to each circle. [*M.U.*]

7. Show that the circles $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x-6y+14=0$ touch each other and find the point of contact. [*M.U.*]

8. Find the equation of the circle which passes through (1, 2) and through the points of intersection of the circles $x^2+y^2-5x-2y+8=0$, $x^2+y^2-3x-y+6=0$.

9. Show that the circles $x^2+y^2-4x-7y+6=0$, $x^2+y^2+3x-14y-1=0$ and $3(x^2+y^2)+2x-35y+4=0$, have a common radical axis.

10. Find the equation of the circle whose diameter is the common chord of the circles $x^2+y^2+2x+3y+1=0$ and $x^2+y^2+4x+3y+2=0$. [*N.U.*]

11. Find the length of the common chord of the circles

(i) $x^2+y^2-12x+16y-69=0$,
$x^2+y^2-9x+12y-59=0$.

(ii) $(x-p)^2+(y-q)^2=r^2$, $(x-q)^2+(y-p)^2=r^2$.

12. Find the points of intersection of the two circles
$$x^2+y^2-2x+4y-4=0,\ x^2+y^2+4x-2y-4=0.$$

13. Show that the lines joining the points of intersection of the circles $x^2+y^2-14x+2y+25=0$, $x^2+y^2-7x+y=0$ to the origin are at right angles.

14. If $lx+my+n=0$ is a tangent to the circle $x^2+y^2+2gx+2fy+c=0$, then it is also a tangent to the circle $x^2+y^2+2gx+2fy+c+k(lx+my+n)=0$ for all values of k.

15. Show that the two circles $x^2+y^2+4x-6y-12=0$, $x^2+y^2-2x+2y-23=0$ cut at an angle of $60°$.

[*If ϕ be the angle between the tangents at the common point, r_1, r_2 the radii, d, the distance between the centres, then $2r_1r_2 \cos\phi = r_1^2+r_2^2-d^2$.*]

16. Find the equation of the circle which cuts orthogonally the three circles :

(i) $x^2+y^2=16,\quad x^2+y^2-14x+40=0,$
$x^2+y^2-12y+32=0.$

(ii) $x^2+y^2-2x+3y-7=0,\ x^2+y^2+5x-5y+9=0,$
$x^2+y^2+7x-9y+29=0.$

(iii) $x^2+y^2=a^2,\ (x-c)^2+y^2=a^2,\ x^2+(y-b)^2=a^2.$
[*C. U. 1939*]

17. Find the condition that the circle $x^2+y^2+2g_1x+2f_1y+c_1=0$ shall bisect the circumference of the circle $x^2+y^2+2g_2x+2f_2y+c_2=0$.

[*The radical axis passes through the centre of the 2nd circle.*]

18. Find the common tangents of the circles

(i) $x^2+y^2-22x+4y+100=0$;
$x^2+y^2+22x-4y-100=0.$ [*C. U. 1940*]

(ii) $x^2+y^2+4x+2y-4=0,\ x^2+y^2-4x-2y+4=0.$
[*C. U.*]

19. If the four points in which the two circles $x^2+y^2+ax+by+c=0$, $x^2+y^2+a'x+b'y+c'=0$ are intersected by the lines $Ax+By+C=0$, $A'x+B'y+C'=0$, lie on another circle, then

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}=0. \qquad [C.\ U.]$$

[*Express the condition that the circles* $S+\lambda L=0$, $S'+\lambda' L'=0$ *are identical.*]

20. Show that the locus of a point which moves so that the tangents from it to two given circles are in a constant ratio is a co-axial circle.

21. Find the radical axis of the co-axial system of circles whose limiting points are $(-1, 2)$ and $(2, 3)$.

[The point circles at the limiting points are $S_1=0$, $S_2=0$, where $S_1 \equiv (x+1)^2+(y-2)^2$ and $S_2 \equiv (x-2)^2+(y-3)^2$; so the radical axis is $S_1-S_2=0$.]

22. Ascertain the limiting points of the co-axial set defined by the pair of circles :

(i) $x^2+y^2-20x-16y+41=0$,
$x^2+y^2-30x-24y+41=0$.

(ii) $x^2+y^2-6x-2y+4=0$,
$3(x^2+y^2)-16x-4y+12=0$.

CHAPTER VIII

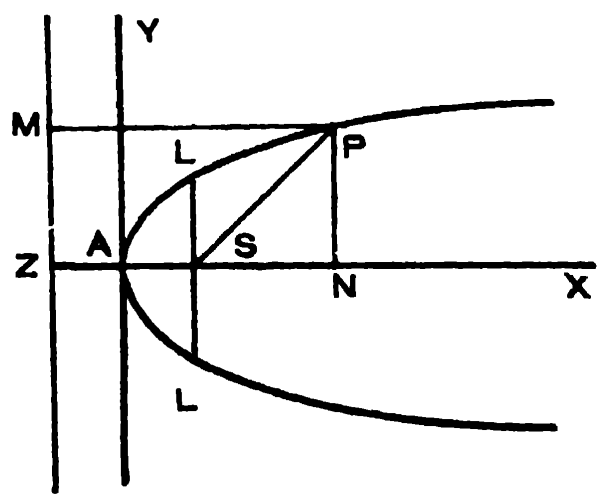
PARABOLA

8·1. Def. A *parabola* is defined as the locus of a point which moves so that its distance from a fixed point called the *Focus* is equal to its perpendicular distance from a fixed line called the *Directrix*. The line passing through the focus and perpendicular to the directrix is called the *Axis*. The point midway between focus and directrix is called the *Vertex*.

8·2. Standard Equation.

To find the equation of a parabola, given its focus and directrix.

Let S be the focus and MZ its directrix. From S draw SZ perp. to MZ. Bisect SZ at A. Let $SA = AZ = a$. Then A is the vertex and ASN, the axis of the parabola.



Let us take A (the vertex) as the origin, ASX (the axes) as the x-axis and a line AY perp. to ASX as the y-axis.

Then the co-ordinates of the focus S are $(a, 0)$ and the equation of the directrix MZ is

$$x + a = 0.$$

Let (x, y) be the co-ordinates of any point P on the parabola. Join SP and draw PM perp. to MZ.

From the definition of the parabola, $SP = PM$.

$$\therefore \qquad SP^2 = PM^2,$$

$$\therefore \qquad (x-a)^2 + y^2 = (x+a)^2,$$

$$i.e. \quad x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2.$$

$$\therefore \qquad \mathbf{y^2 = 4ax.}$$

This being the relation between the co-ordinates of any point on the parabola is the equation of the parabola. This is called *Standard Equation*. Throughout this chapter, this form of the equation will be used, unless otherwise stated.

Cor. 1. The double ordinate LSL' passing through the focus and perp. to the axis is called *Latus rectum*. Obviously we have SL = distance of L from $MZ = SZ = 2a$.

Hence Latus Rectum $= 4a$.

Cor. 2. Two parabolas are said to be *equal* when they have the same latus rectum.

Note. The following results which are of frequent occurrence are collected together for ready reference

(i) **Co-ordinates of vertex (0, 0)** ;

(ii) **Co-ordinates of the focus (a, 0)** ;

(iii) **Equation of the directrix $\mathbf{x + a = 0}$** ;

(iv) **Equation of the axis $\mathbf{y = 0}$** ;

(v) **Length of the latus rectum $\mathbf{= 4a}$.**

Peculiarities of the curve

The parabola is symmetrical with regard to the x-axis *i.e.* the axis of the curve, since when $-y$ is substituted for y, the form of the equation is unchanged.

Again if x be not positive, there will be no real value of y. This shows that no part of the curve lies to the left of the y-axis.

Lastly x can have any positive value however great and hence also y. Thus the curve consists of two branches, which are really continuations of each other, both extending to infinity, one lying above and the other below the axis.

8·3. Different Forms.

(A) If the focus be situated on the left side of the directrix, the equation of the parabola with the vertex as origin and the axis as x-axis, is

$$y^2 = -4ax, \quad \text{(Fig. (i))}$$

since in this case the focus S is $(-a, 0)$. The latus rectum however is a positive quantity and equal to $4a$.

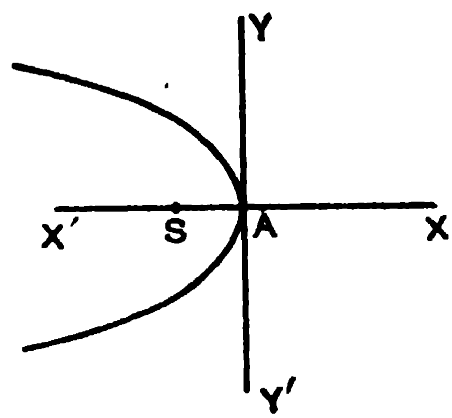


Fig. (i)

(B) Again the vertex being the origin, if the axis of the parabola is taken as y-axis, the equation of the parabola is

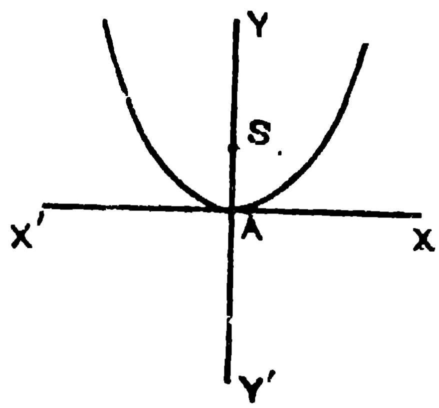


Fig. (ii)

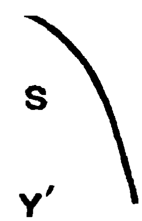


Fig. (iii)

$$x^2 = 4ay, \quad \text{(Fig. (ii))}$$

$$x^2 = -4ay, \quad \text{(Fig. (iii))}$$

according as the focus is above or below the x-axis. The latus rectum in each case is $4a$.

8·4. Particular Cases.

If the focus S is taken as the origin, the axis as x-axis, and a line through S perp. to the axis as y-axis, then the co-ordinates of S are $(0, 0)$ and equation of MZ is $x + 2a = 0$.

Hence from the property $SP=PM$, i.e. $SP^2=PM^2$, we have

$$x^2+y^2=(x+2a)^2,$$

$$\text{i.e.} \quad x^2+y^2=x^2+4ax+4a^2,$$

$$\text{i.e.} \quad y^2=4a(x+a).$$

This is often spoken as the *equation of the parabola with focus as origin.*

Similarly, if Z be taken as origin, and ZX as x-axis and ZM as y-axis, the equation of the parabola is

$$(x-2a)^2+y^2=x^2,$$

$$\text{or} \quad y^2=4a(x-a).$$

8·5. General Equation.

To find the equation of a parabola, given its focus and directrix.

Let (α, β) be the focus S and $lx+my+n=0$ be the directrix (See Fig. of Art. 8·2). Let (x, y) be any point P on the parabola. Join SP and draw PM perp. to the directrix.

$$\text{Since} \quad SP=PM.$$

$$\therefore \quad SP^2=PM^2.$$

$$\therefore \quad (x-\alpha)^2+(y-\beta)^2=\left\{\frac{lx+my+n}{\sqrt{l^2+m^2}}\right\}^2$$

This on simplification reduces to

$$m^2x^2-2lmxy+l^2y^2-2x\{(l^2+m^2)\alpha+ln\}$$
$$-2y\{(l^2+m^2)\beta+mn\}+(l^2+m^2)(\alpha^2+\beta^2)-n^2=0.$$

From above, we see that the *terms of the second degree i.e.* $m^2x^2-2lmxy+l^2y^2$ *form a perfect square* $(mx-ly)^2$. This is a *characteristic property of the equation of a parabola.*

8·6. Locus of (i) $\mathbf{y=ax^2+bx+c}$,
(ii) $\mathbf{x=ay^2+by+c}$.

The first equation can be written as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \frac{y}{a},$$

or $$\left(x + \frac{b}{2a}\right)^2 = \frac{y}{a} + \frac{b^2}{4a^2} - \frac{c}{a} = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right).$$

Transferring the origin to $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$, the above equation transforms into

$$x^2 = \frac{1}{a}y$$

which obviously represents a parabola of which vertex is the new origin and axis is the new y-axis. Hence *the equation $y = ax^2 + bx + c$ always represents a parabola having its axis parallel to y-axis, vertex* $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$ and *latus rectum* $= \frac{1}{a}$.

Similarly, the second equation can be written as

$$\left(y + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(x + \frac{b^2 - 4ac}{4a}\right);$$

hence transferring the origin to $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$, the equation transforms into

$$y^2 = \frac{1}{a}x.$$

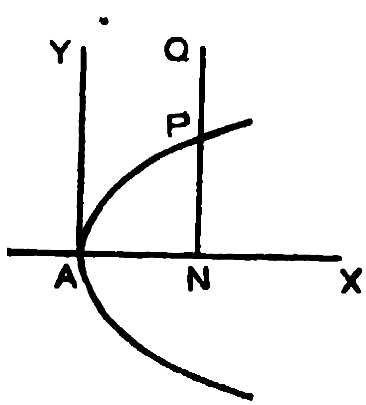
which obviously represents a parabola whose vertex is the new origin and axis is the new x-axis. Hence *the equation $x = ay + by + c$ represents a parabola having its axis parallel to the x-axis*, vertex $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$ and *latus rectum* $= 1/a$.

Cor. As a particular case, the equations $y = ax^2 + bx$ and $x = ay^2 + by$ represent parabolas having their axis respectively parallel to the y-axis and x-axis.

Note. It is clear from above that *if any equation be linear in one of the variables say x or y and quadratic in the other*, then *the equation always represents a parabola whose axis is parallel to x-axis or y-axis according as the equation is linear in x or in y.*

8·7. Position of a point in relation to a parabola.

The point (x_1, y_1) *lies outside, upon or inside the parabola* $y^2 = 4ax$ *according as* $y_1^2 - 4ax_1 > = or < 0$.



Let Q be the point (x_1, y_1); draw QN perp. to the axis cutting the parabola in P. Then the abscissa of P is x_1 and its ordniate is PN.

Hence $\qquad PN^2 = 4ax_1.$

Now, Q lies outside, upon or inside the parabola

if $\qquad QN^2 > = \text{ or } < PN^2,$

i.e. if $\qquad y_1^2 > = \text{ or } < 4ax_1,$

i.e. if $\qquad y_1^2 - 4ax_1 > = \text{ or } < 0.$

8·8. Focal Distance of a point.

Let (x_1, y_1) be the co-ordinates of a point P. [*See fig. of Art. 8·2*] Then its *focal distance*

$$= SP = PM = ZN = ZA + AN = \mathbf{x_1 + a}.$$

8·9. Intersection of a line and a parabola.

To find the points of intersection of the line

$$y = mx + c, \qquad \dots \qquad (1)$$

with the parabola $y^2 = 4ax. \qquad \dots \qquad \dots \qquad (2)$

Substituting the value of y from (1) in (2), we get the equation for the abscissæ of the common points of (1) and (2), *viz.* $\qquad (mx + c)^2 = 4ax,$

or $\qquad m^2x^2 + 2(mc - 2a)x + c^2 = 0. \qquad \dots \qquad (3)$

Similarly eliminating x between (1) and (2), we get the quadratic for the ordinates of the common points *viz.*

$$my^2 - 4ay + 4ac = 0. \qquad \dots \qquad (4)$$

From (3) or (4), it is clear that a line intersects the parabola in *two* points, real, coincident or imaginary. The points will be real, coincident or imaginary according as the roots of either quadratic say (3) are real, coincident or imaginary

i.e. according as $4(mc - 2a)^2 - 4m^2c^2 > = \text{ or } < 0$

i.e. ,, $a(a - mc) > = \text{ or } < 0,$

i.e. ,, $c < = \text{ or } > \dfrac{a}{m}.$

It should be noted that the ordinates of the points of intersection can also be obtained by substituting two roots say x_1, x_2 of (3) in (1), thus $y_1 = mx_1 + c$, $y_2 = mx_2 + c$.

8·10. Tangent at a point.

To find the equation to the tangent at the point (x_1, y_1) *of the parabola* $y^2 = 4ax$.

Let P be the given point (x_1, y_1) and $Q(x_2, y_2)$ be a point on the parabola very close to P.

The equation to PQ is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \qquad \dots \qquad (1)$$

Since P, Q lie on $y^2 = 4ax$.

$$\therefore \quad y_1^2 = 4ax_1, \text{ and } y_2^2 = 4ax_2.$$

$$\therefore \quad y_1^2 - y_2^2 = 4a(x_1 - x_2).$$

$$\therefore \quad \frac{y_1 - y_2}{x_1 - x_2} = \frac{4a}{y_1 + y_2}. \qquad \dots \qquad (2)$$

$\therefore$ by (2), *the equation of the chord BQ* becomes

$$y - y_1 = \frac{4a}{y_1 + y_2}(x - x_1). \qquad \dots \qquad (3)$$

When Q comes to coincide with P, the secant PQ becomes the tangent at P; in this case $y_2 = y_1$. Hence putting $y_2 = y_1$ in (3) the equation of the tangent at P is

$$y - y_1 = \frac{4a}{2y_1}(x - x_1),$$

or $$yy_1 - y_1^2 = 2ax - 2ax_1,$$

or $$yy_1 - 4ax_1 = 2ax - 2ax_1.$$

$$\therefore \quad \mathbf{yy_1 = 2a(x + x_1).}$$

Cor. Tangent at the vertex (0, 0) is x = 0.

8·11. Condition of tangency of a line.

To find the condition that the line $y = mx + c$ $\cdots$ (1) *shall tauch the parabola* $y^2 = 4ax$. $\cdots$ (2).

The abscissæ of the points of intersection of (1) and (2) are given by the roots of

$$(mx + c)^2 = 4ax,$$

i.e. of $m^2x^2 + 2(mc - 2a)x + c^2 = 0$. $\quad\ldots\quad$ (3)

The line (1) will touch (2) if the two points of intersection are coincident *i.e.* if the two roots of (3) are equal,

i.e. if $$4(mc - 2a)^2 = 4m^2c^2,$$

i.e. if $$-4amc + 4a^2 = 0,$$

i.e. if $$\mathbf{c = \frac{a}{m}}.$$

Hence the line $\mathbf{y = mx + \frac{a}{m}}$ *is always a tangent to the parabola* $y^2 = 4ax$, *whatever be the value of* m. Here m is the tangent of the angle which the tangent makes with the x-axis.

Also any of the equal roots of (3), gives the abscissa of the point of contact. Thus,

$$x_1 = \frac{2a - mc}{m^2} = \frac{a}{m^2}, \quad \text{since } c = \frac{a}{m}$$

Again, $y_1 = mx_1 + c = m \cdot \frac{a}{m^2} + \frac{a}{m} = \frac{2a}{m}$.

$\therefore$ co-ordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Alternative Method :

Let (x_1, y_1) be the point where the line (1) touches the parabola (2). The equation of the tangent at (x_1, y_1) to (2) is $yy_1 = 2ax + 2ax_1$ ... ... (3)

$\therefore$ (1) and (3) must represent the same line.

$$\therefore \quad \frac{y_1}{1} = \frac{2a}{m} = \frac{2ax_1}{c}$$

$$\therefore \quad y_1 = \frac{2a}{m}, \quad x_1 = \frac{c}{m}. \quad \cdots \quad \cdots \quad (4)$$

Since $y_1^2 = 4ax_1$. $\therefore \frac{4a^2}{m^2} = \frac{4ac}{m}$.

$\therefore$ $c = a/m$ is the reqd. condition of tangency,

Substituting the value of c in (4), we get the co-ordinates of the point of contact *viz.* $(a/m^2, 2a/m)$.

8·12. Normal at a point.

To find the equation of the normal at the point (x_1, y_1) to the parabola $y^2 = 4ax$.

The tangent at (x_1, y_1) is $y = \frac{2a}{y_1}x + \frac{2a}{y_1}x_1$. ... (1)

Any line through (x_1, y_1) is $y - y_1 = m(x - x_1)$ (2)

If (2) be normal at (x_1, y_1), it must be perp. to (1).

$$\therefore \quad m \times \frac{2a}{y_1} = -1, \quad i.e. \quad m = -\frac{y_1}{2a}.$$

$\therefore$ the reqd. equation of the normal is

$$\mathbf{y - y_1 = -\frac{y_1}{2a}(x - x_1)} \quad \cdots \quad \cdots \quad (1)$$

Alternative form (m-form).

Let us put $-\dfrac{y_1}{2a}=m \quad \therefore \quad y_1=-2am.$

Hence $x_1=\dfrac{y_1^2}{4a}=\dfrac{4a^2m^2}{4a}=am^2.$

Substituting these values of x_1, y_1 in (1), the equation of the normal at $(am^2, -2am)$ is

$$y+2am=m(x-am^2),$$

$$\text{i.e.,} \qquad \mathbf{y=mx-2am-am^3.}$$

Note. Here m is the tangent of the angle which the normal at any point makes with the x-axis.

8·13. Number of tangents from a point.

To show that from any point there can be drawn two tangents to a parabola.

Let (x_1, y_1) be the point and the equation of the parabola be $y^2=4ax.$ ... ... (1)

The equation of any tangent to the parabola (1) is

$$y=mx+\frac{a}{m}, \quad \cdots \quad \cdots \quad (2)$$

where m is a variable parameter.

Let us choose the parameter m so that (2) may pass through (x_1, y_1).

$$\therefore \quad y_1=mx_1+\frac{a}{m},$$

$$\text{or,} \quad m^2x_1-my_1+a=0. \quad \cdots \quad (3)$$

The equation being a quadratic in m gives two values of m, each of which being substituted in (2) leads to a tangent through (x_1, y_1).

Evidently, *the two tangents from* (x_1, y_1) will be ***real***, ***coincident***, or ***imaginary*** according as the roots of (3) are real, coincident or imaginary ; that is

$$\text{according as} \quad y_1^2-4ax_1 > = \text{or} < 0,$$

i.e. according as the point (x_1, y_1) *is outside, upon* or *inside the parabola.* [Art. 8·7]

8·14. Pole and Polar.

The Pole and Polar with respect to a parabola are defined exactly in the same way as in the case of the circle.

[*Art.* 6·12]

(A) *To find the polar of the point* (x_1, y_1) *w. r. t. the parabola* $y^2 = 4ax$.

If the points of contact be $A(x_2, y_2)$, $B(x_3, y_3)$ the tangents at these points are

$$yy_2 = 2a(x + x_2).$$
$$yy_3 = 2a(x + x_3).$$

$\because$ both these tangents pass through $P(x_1, y_1)$, we have

$$y_1 y_2 = 2a(x_1 + x_2).$$
$$y_1 y_3 = 2a(x_1 + x_3).$$

These relations show that the two points A, B are situated on the line

$$\mathbf{yy_1 = 2a(x + x_1)},$$

which is therefore the reqd. equation of the polar of the point (x_1, y_1).

(B) *To find the pole of the line* $lx + my + n = 0$ *with respect to the parabola* $y^2 = 4ax$.

Let (x_1, y_1) be the required pole.

The polar of (x_1, y_1) with respect to $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1),$$

$$i.e. \quad 2ax - yy_1 + 2ax_1 = 0,$$

This must be identical with

$$lx + my + n = 0.$$

$$\therefore \quad \frac{2a}{l} = \frac{-y_1}{m} = \frac{2ax_1}{n}.$$

$$\therefore \quad x_1 = \frac{n}{l}, \; y_1 = -\frac{2am}{l}.$$

8·15. Properties of Pole and Polar.

(i) *If the polar of a point. P w. r. t. a parabola passes through Q, the polar of Q passes through P.*

Let the co-ordinates of P, Q be (x_1, y_1), (x_2, y_2) and let the parabola be $y^2=4ax$.

The polar of P is $yy_1=2a(x+x_1)$. $\cdots$ (1)

The polar of Q is $yy_2=2a(x+x_2)$. $\cdots$ (2)

If (1) passes through Q, then

$$y_2y_1=2a(x_2+x_1)$$

which is exactly the condition that (2) passes through P.

(ii) *The point of intersection of any two lines is the pole of the line joining the poles of the lines.*

The proof is the same as that given in Art. 6·13.

(ii) *If the pole of the line $lx+my+n=0$ w. r. t a parabola lies on the line $l'x+m'y+n'=0$, then the pole of the line $l'x+m'y+n'=0$* lies on $lx+mx+n=0$.

Pole of $lx+my+n=0$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$. $\cdots$ (1)

Pole of $l'x+m'y+n'=0$ is $\left(\frac{n'}{l'}, -\frac{2am'}{l'}\right)$. $\cdots$ (2)

Since the pole (1) lies on $l'x+m'y+n'=0$.

$$\therefore \quad l'\frac{n}{l}-m'.\frac{2am}{l}+n'=0,$$

$$i.e. \qquad ln'-2amn'+l'n=0,$$

which is exactly the condition that the pole (2) lies on

$$lx+my+n=0.$$

8·16. Chord of contact.

From the definition of the polar of a point, it is clear that the polar of a point is nothing but the chord of contact

produced infinitely both ways. Hence *the equation of the chord of contact of the tangents drawn from a point* (x_1, y_1) is to be obtained exactly in the same way as in the case of the polar of the point (Art. 8·14) and its equation is

$$\mathbf{yy_1 = 2a(x+x_1).}$$

Note. It should be noted that the *equation of the tangent* at (x_1, y_1), *equation of the polar* of the point (x_1, y_1) and the *equation of the chord of contact of* tangents drawn from (x_1, y_1) are *identical* in form.

8·17. Illustrative Examples.

Ex. 1. *Find the equation of the parabola whose focus is* $(-8, -2)$ *and directrix is* $y = 2x - 9$.

Let $P(x, y)$ be any point on the parabola. Since the distance of any point on a parabola from the focus is equal to its distance from the directrix,

$$\therefore \qquad \sqrt{(x+8)^2 + (y+2)^2} = \frac{2x - y - 9}{\sqrt{5}}.$$

Squaring, $\quad 5\{(x+8)^2 + (y+2)^2\} = (2x - y - 9)^2$,

or $\qquad x^2 + 4xy + 4y^2 + 116x + 2y + 259 = 0.$

Ex. 2. *Find the directrix of the parabola whose focus is* (3, 4) *and vertex* (0, 0).

Since vertex is midway between the focus S and Z (the point of intersection of the axis and directrix).

$$\therefore \quad Z \text{ is } (-3, -4).$$

Now the axis of the parabola being the join of (0, 0) and (3, 4), its 'm' is $\frac{4}{3}$.

The directrix being the line through Z, perp. to the axes, its equation is

$$y + 4 = -\tfrac{3}{4}(x+3),$$

or $$3x + 4y + 25 = 0.$$

Ex. 3. *Find the length of the chord intercepted by* $y^2 = 4ax$ on $y = mx + c$.

Let (x_1, y_1), (x_2, y_2) be the common points of intersection.

The abscissæ of the common points of the parabola and the line are roots of

$$(mx+c)^2=4ax,$$

or $$m^2x^2+2x(mc-2a)+c^2=0. \qquad \dots \qquad (1)$$

Now, from (1), $$(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$

$$=\frac{4(mc-2a)^2}{m^4}-4\frac{c^2}{m^2}=\frac{16a(-mc)}{m^4}.$$

And $$y_1=mx_1+c,\ y_2=mx_2+c.$$

$\therefore$ $$y_1-y_2=m(x_1-x_2).$$

$\therefore$ If D be the reqd. length

$$D^2=(x_1-x_2)^2+(y_1-y_2)^2=(x_1-x_2)^2+m^2(x_1-x_2)^2$$

$$=\frac{16a(1+m^2)}{m^4}\cdot(a-mc).$$

whence D can be obtained.

Ex. 4. *Show that the locus of the foot of the perpendicular from the focus upon any tangent of $y^2=4ax$ is the tangent at the vertex.*

Let $y=mx+\dfrac{a}{m}$ ... (1) be any tangent.

The line through focus $(a, 0)$, perp. to (1) is

$$y=-\frac{1}{m}(x-a),$$

or $$y=-\frac{1}{m}x+\frac{a}{m} \qquad \dots \qquad (2)$$

Subtracting (2) from (1), $\left(m+\dfrac{1}{m}\right)x=0.$

Hence the reqd. locus is $x=0$, the tangent at the vertex. This called the *positive pedal* of the parabola *w. r. t.* the focus.

Ex. 5. *Two equal parabolas have the same vertex and their axes are at right angles. Prove that the common tangent touches each at the end of its latus rectum.*

[*C. U. 1935, '39, '44*]

Let the equations of the parabolas be

$$y^2=4ax. \qquad \dots \qquad (1)$$

$$x^2=4ay. \qquad \dots \qquad (2)$$

Now, $y=mx+\dfrac{a}{m}$ is a tangente to (1). If it also be a tangent to (2), the the quadratic giving the abscissæ of the points of intersection *viz.*

$$x^2=4a\left(mx+\frac{a}{m}\right),$$

or $$mx^2-4am^2x-4a^2=0$$

must have equal roots, the condition for which is $16a^2m^4=-16a^2m$ *i.e.* $m^3=-1$ *i.e.* $m=-1$.

Hence the equation of the common tangent is

$$x+y+a=0. \quad \dots \quad (3)$$

Suppose this tangent touches (1) at (x_1, y_1). The tangent at (x_1, y_1) is

$$\therefore \quad 2ax-yy_1+2ax_1=0, \quad \dots \quad (4)$$

Comparing (3) and (4), $2a=-y_1=\dfrac{2ax_1}{a}$.

$\therefore$ $x_1=a$, $y_1=-2a$, which point is obviously one end of the latus rectum of (1). Similarly it can be shown that the common tangent touchs (2) at one end $(-2a, a)$ of its latus rectum.

Examples VIII(A)

1. Find the equation of the parabola

(i) whose focus is $(1, -1)$ and directrix is $x+y+7=0$;

(ii) whose focus is $(5, 3)$ and directrix is $3x-4y+1=0$;

(iii) whose focus is $(0, 0)$ and tangent at the vertex is $x-y+1=0$;

(iv) whose vertex is $(-2, 2)$ and directrix is $x+y-4=0$;

(v) whose focus is $(-6, -6)$ and vertex is $(-2, 2)$;

(vi) whose vertex and focus are on the x-axis at distances a and b $(b > a)$ from the origin respectively.

2. Fine the vertex and axis of the parabola whose focus is (3, 4) and directrix is $3x+4y+25=0$.

3. Find the vertex, focus, axes, directrix and latus rectum of each of the parabolas

(i) $4y^2-20x-8y+39=0$,

(ii) $5x^2+15x-10y-4=0$.

4. Find the equation to the parabola whose axis is parallel to y-axis and which passes through the points (0, 2), (1, −3), (4, 6).

5. Show that the equation of the parabola, which passes through the points (x_1, y_1), (x_2, y_2), (x_3, y_3) and whose axis is parallel to y-axis, is

$$\begin{vmatrix} y & x^2 & x & 1 \\ y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \end{vmatrix}=0.$$

6. Find the positions of the following points *w. r. t.* the parabola $y^2=4ax$:—

(2, −1), (1, 4), (−3, 4).

7. Find the vertex and latus rectum of the parabola

$$x=u\cos a.\ t,$$
$$y=u\sin a.\ t-\tfrac{1}{2}gl^2,$$

where t is a variable parameter.

[*Eliminate t between the equations ; then it will reduce to the form* $y=ax^2+bx$].

8. Show that the equation of the chord of the parabola $y^2=4ax$ through the points (x_1, y_1), (x_2, y_2) is

$$(y-y_1)(y-y_2)=y^2-4ax,$$

and hence deduce the equation of the tangent at (x_1, y_1).

9. The co-ordinates of the ends of a focal chord of the parabola $y^2=4ax$ are (x_1, y_1), (x_2, y_2). Prove that

$$y_1y_2+4x_1x_2=0.$$

10. (i) Show that the line $lx+my+n=0$ touches the parabola $y^2=4ax$ if $am^2-nl=0$.

(ii) Show that the line $x\cos\alpha+y\sin\alpha-p=0$ touches the parabola $y^2=4ax$ if $p\cos\alpha+a\sin^2\alpha=0$ and the point of contact is $(a\tan^2\alpha,\ -2a\tan\alpha)$.

11. Show that the line $y=m(x+a)+\dfrac{a}{m}$ touches the parabola $y^2=4a(x+a)$ for all values of m.

12. Find the equations of the tangents and normals at the ends of the latus rectum of $y^2=4ax$.

13. Find the equation of the tangent to $y^2=8x$ which is

(i) parallel to $2x-3y+1=0$,

(ii) perpendicular to $x+2y+7=0$.

14. Find the point of the parabola $y^2=4ax$ at which the normal is inclined at $30°$ to axis. [*C. U.*]

15. Prove that the tangent and the normal at any point of a parabola meet the axis at points which are equidistant from the focus.

16. Find the angle at which the two parabolas $y^2=4ax$, $x^2=4by$ cut one another. [*C. U.*]

17. Prove that two parabolas which have the same focus and whose axes are in opposite directions cut orthogonally.

[*Take the common focus as origin and the common axis as x-axis.*]

18. Establish analytically the following results

(i) The subtangent at any point of a parabola is bisected at the vertex.

(ii) The subnormal at any point of a parabola is constant and equal to semi-latus rectum.

(iii) If SY be perpendicular from the focus upon the tangent at any point P of a parabola, then $SY^2=AS.SP$. Hence deduce $p^2=ar$.

(iv) If the normal drawn at P intersects the axes at G, show that $PG^2=4AS.SP$.

19. The normal to the parabola $y^2 = 4ax$ at $(am_1^2, -2am_1)$ meets the curve again at $(am_2^2, -2am_2)$. Show that $m_1^2 + m_1m_2 + 2 = 0$. [*C.U.*]

20. Find the locus of the foot of the perpendicular from the vertex on the tangent at any point of $y^2 = 4ax$.

21. Prove that the locus of the poles of a series of parallel lines *w. r. t.* a parabola is a line parallel to the axis.

22. Show that the locus of poles of chords of $y^2 = 4ax$ which subtend a right angle at the vertex is a line perpendicular to the axis.

23. Show that the locus of poles of tangents to $y^2 = 4ax$ *w. r. t.* $y^2 = 4bx$ is $y^2 = 4cx$ where $c = b^2/a$. [*C. U. 1942*]

24. Two equal parabolas A and B, have the same vertex and axis, but have their concavities turned in opposite directions. Prove that the locus of poles *w. r. t.* B of tangents to A is the parabola A. [*C. U. 1939*]

25. Prove that the locus of the poles of tangents to $y^2 + 4ax = 0$ *w. r. t.* $x^2 + y^2 + 2ax - 3a^2 = 0$ is $x^2 + y^2 - 2ax - 3a^2 = 0$.

26. Find the locus of the poles of the normal chord of $y^2 = 4ax$.

27. If a circle be drawn so as always to touch a given line and also a given circle, prove that the locus of its centre is a parabola. [*C. U. 1940*]

[*Take the centre of the given circle as origin and the line through it parallel to the given line as x-axis.*]

28. Prove that the normal chord of a parabola at the point whose ordinate is equal to its abscissa subtends a right angle at the focus. [*C. U. 1940*]

29. Show that the tangents at the extremities of a focal chord of a parabola meet on the directrix. [*C.U. 1943*]

30. Show that the locus of the point of intersection of tangents at the extremities of any chord of $y^2 = 4ax$ which subtends a right angle at the vertex is $x + 4a = 0$.

31. Show that the line $x-2y+5=0$ is a tangent to the parabola $y^2-4x-4y+16=0$.

32. Prove that the line $lx+my+n=0$, touches the parabola $y^2=4a(x-b)$ if $am^2=bl^2+nl$.

33. Find the common tangents of

(i) $y^2=4ax$ and $x^2=4by$;

(ii) $y^2=8ax$ and $x^2+y^2=2a^2$.

34. Show that the point of intersection of two perpendicular lines one of which touches $y^2=4a(x+a)$ and the other $y^2=4b(x+b)$, is on the line $x+a+b=0$ which is the common chord of the two parabolas.

35. A point moves in such a way that its distance from a given line is equal to the length of a tangent drawn from the point to a given circle. Show that the locus of the point is a parabola.

[*Take the centre of the given circle as origin and a line parallel to the given line as x-axis.*]

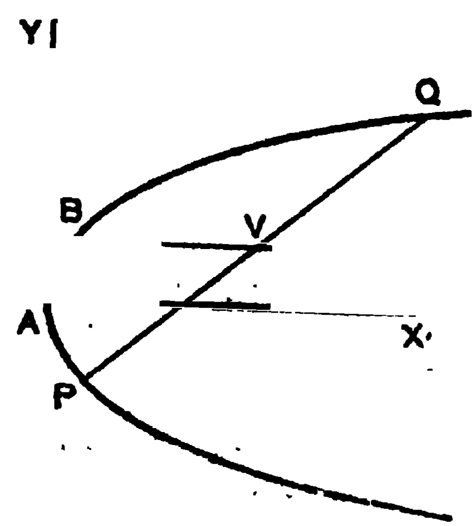
8·18. Diameter.

To find the locus of the middle points of a system of parallel chords of the parabola $y^2=4ax$.

Let us represent any chord PQ of the parallel system in the form

$$y=mx+c, \qquad \cdots \quad (1)$$

where m has a fixed value and c is different for different chords.



Now PQ meets the parabola in points whose ordinaates re given by

$$y^2 = 4a.\frac{y-c}{m}$$

i.e. $my^2-4ay+4ac=0$. $\qquad$ (2)

Let the roots of the equation *i.e.* the ordinates of P, Q be y_1, y_2 and let $(\bar{x}, \bar{y})$ be the co-ordinates of the mid-point V of the chord.

Then, $\bar{y} = \dfrac{y_1 + y_2}{2} = \dfrac{2a}{m}$ from (2)

$= \text{constant}$ (independent of c).

Hence V lies on the line

$$y = \frac{2a}{m} \quad \cdots \quad \cdots \quad (3)$$

which is parallel to the axis of the parabola.

Thus, *the locus of the mid-points of a system of parallel chords of a parabola is a line parallel to the axis of the parabola.*

Cor. 1. The *axis of a parabola is a diameter* which is perpendicular to the chords it bisects.

Def. The locus of the middle points of a system of parallel chords of a parabola is called a *Diameter* and the chords bisected by it are called the double ordinates.

8·19. Equation of a chord in terms of its middle point.

To find the equation of a chord of $y^2 = 4ax$ in terms of its middle point (x_1, y_1).

The equation of any chord through (x_1, y_1) is

$$y - y_1 = m(x - x_1). \quad \cdots \quad \cdots \quad (1)$$

Let (x', y'), (x'', y'') be the points of intersection of this chord and the parabola. The ordinates of the points of intersection of the chord and the parabola are given by the roots of the quadratic in y, obtained by eliminating x between (1) and $y^2 = 4ax$ *i.e.* by

$$y^2 = 4a\left(\frac{y - y_1 + mx_1}{m}\right),$$

or $\quad my^2 - 4ay + 4ay_1 - 4amx_1 = 0.$

Now, $$y_1 = \frac{y' + y''}{2} = \frac{4a}{2m} = \frac{2a}{m}.$$

$\therefore \quad m = 2a/y_1.$

Substituting this value of m in (1), the required equation of the chord is

$$(\mathbf{y} - \mathbf{y}_1)\mathbf{y}_1 = \mathbf{2a}(\mathbf{x} - \mathbf{x}_1).$$

8·20. Pair of tangents from a point.

To find the equation of pair of tangents from (x_1, y_1) to the parabola $y^2 = 4ax$.

The equation of any line through (x_1, y_1) is

$$y - y_1 = m(x - x_1), \quad \cdots \quad \cdots \quad (1)$$

or $\quad y = mx + y_1 - mx_1.$

If it touches $y^2 = 4ax$,

then, $\quad y_1 - mx_1 = \dfrac{a}{m}$. [Art. 8·11] $\quad \cdots \quad (2)$

Now eliminating m between (1) and (2) *i.e.* substituting the value of m from (1) in (2), we get the equation of the pair of tangents as

$$y_1 - \frac{y - y_1}{x - x_1}x_1 = \frac{a(x - x_1)}{y - y_1},$$

or $\quad (xy_1 - yx_1)(y - y_1) = a(x - x_1)^2.$

This when simplified can be put in the form

$$(y^2 - 4ax)(y_1{}^2 - 4ax_1) = \{yy_1 - 2a(x + x_1)\}^2.$$

8·21. Parametric Representation.

Since $x = at^2$, $y = 2at$ satisfy the equation of the parabola $y^2 = 4ax$ whatever be the value of t, co-ordinates of any point on the above parabola can be represented by

$$\mathbf{x} = \mathbf{at}^2,$$
$$\mathbf{y} = \mathbf{2at}.$$

By eliminating t, we get the equation in x, y ; the point whose co-ordinates are $(at^2, 2at)$ is, for sake of brevity called **the point "t".**

8·22. Equation of the chord in parametric co-ordinates.

To find the equation of the chord of the parabola $y^2 = 4ax$ *joining the points* $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.

The *equation of the line joining the points*, is

$$\frac{x - at_1^2}{at_1^2 - at_2^2} = \frac{y - 2at_1}{2at_1 - 2at_2},$$

$$\text{or} \quad \frac{x - at_1^2}{t_1 + t_2} = \frac{y - 2at_1}{2},$$

$$\text{or} \quad 2x - 2at_1^2 = (t_1 + t_2)(y - 2at_1),$$

$$\text{or} \quad \mathbf{y(t_1 + t_2) = 2x + 2at_1t_2}. \quad \cdots \quad (1)$$

(A) Equation of the tangent.

Putting $t_1 = t_2 = t$, in (1), the *equation of the tangent at 't'* is

$$ty = x + at^2. \quad \cdots \quad \cdots \quad (2)$$

(B) Equation of the normal.

Hence the *equation of the normal at 't'* is

$$y - 2at = -t(x - at^2),$$

$$\text{or} \quad y + tx = 2at + at^3. \quad \cdots \quad (3)$$

(C) Point of intersection of tangents.

The *point of intersection of tangents* at $'t'_1$ and $'t'_2$ *i.e.* of $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$ is $\{at_1t_2, a(t_1 + t_2)\}$.

(D) Geometrical meaning of t.

Since the equation of the tangent at 't' is

$$y = \frac{1}{t}x + at,$$

it follows that $\frac{1}{t} = \tan\theta$ *i.e.* $t = \cot\theta$, where θ is the angle which the tangent makes with the x-axis. Hence the *geometrical interpretation of 't'* is that it is the cotangent of the angle which the tangent at the point makes with the x-axis.

(E) **Focal chord.**

The equation of the chord joining the points 't'$_1$ and 't'$_2$ is

$$y(t_1+t_2)=2x+2at_1t_2.$$

If it passes through the focus $(a, 0)$.

$$2a+2at_1t_2=0.$$

$$\therefore \; t_1t_2=-1, \text{ or } t_2=-1/t_1.$$

$\therefore$ *for a focal chord whose extremities are* 't'$_1$, 't'$_2$, we have

$$\mathbf{t_1t_2=-1.}$$

The ends of a focal chord can therefore be represented by

$$(at^2, 2at), \; (a/t^2, -2a/t).$$

8·23. An Important Locus.

The locus of the point of intersection of a pair of perpendicular tangents to a parabola is the directrix.

Any tangent to the parabola is

$$y=mx+\frac{a}{m}.$$

If it pass through (x_1, y_1)

$$y_1=mx_1+\frac{a}{m},$$

$$\text{or} \quad m^2x_1-my_1+a=0. \qquad \cdots \quad (1)$$

If m_1, m_2 be the roots of (1), then $m_1m_2=a/x_1$, $\cdots$ (2) and the equations of the two tangents are $y=m_1x+\dfrac{a}{m_1}$ and $y=m_2x+\dfrac{a}{m_2}$. Since these tangents are perpendicular, $m_1m_2=-1$ and hence from (2), $-a/x_1=1$, *i.e.* $x_1+a=0$.
$\therefore$ locus of (x_1, y_1) *i.e.* of the point of intersection of the perpendicular tangents is the line $x+a=0$, which is the directrix.

Cor. The chord of contact of the two perp. tangents drawn from (x_1, y_1) *i.e.* from $(-a, y_1)$ is $yy_1 = 2a(x-a)$ which being satisfied by $(a, 0)$ passes through the focus. Thus, *the locus of the intersection of a pair of orthogonal tangents to a parabola is the directrix and the associated chord of contact passes through a fixed point viz. the focus.*

Alternative method.

Any tangent to the parabola is

$$y = mx + \frac{a}{m}, \quad \cdots \quad \cdots \quad (1)$$

and a perpendicular tangent is

$$y = -\frac{1}{m}x - am. \quad \cdots \quad \cdots \quad (2)$$

Let (x_1, y_1) be their point of intersection.

$$\therefore \quad y_1 = mx_1 + \frac{a}{m}, \quad \cdots \quad \cdots \quad (3)$$

$$y_1 = -\frac{1}{m}x_1 - am. \quad \cdots \quad \cdots \quad (4)$$

Subtracting (4) from (3),

$$\left(m + \frac{1}{m}\right)x_1 + \left(m + \frac{1}{m}\right)a = 0$$

$$i.e. \quad x_1 + a = 0$$

$\therefore$ locus of (x_1, y_1) is the line $x + a = 0$ which is the directrix.

8·24. Number of normals from a point.

To prove that in general three normals can be drawn from a point to a parabola and the algebraic sum of the ordinates of the feet of these three normals is zero.

The line $$y = mx - 2am - am^3 \quad \cdots \quad (1)$$

is a normal to the parabola at the point

$$(am^2, \ -2am). \quad \cdots \quad (2)$$

If this normal passes through a fixed point (h, k) then

$$k = mh - 2am - am^3,$$

$$\text{or} \quad am^3 + (2a - h)m + k = 0. \quad \cdots \quad (3)$$

The equation, being of the third degree in m, has three roots (real or imaginary), corresponding to each of which there will be a normal.

Let m_1, m_2, m_3 be the roots of (3),

then $\qquad m_1 + m_2 + m_3 = 0.$ [Art. 1·10]

If y_1, y_2, y_3 be the ordinates of the three feet, then by (2), we get

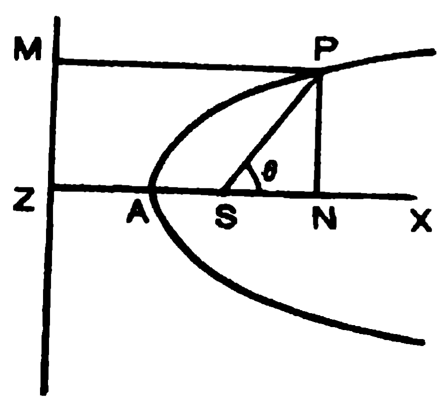
$$y_1 + y_2 + y_3 = -2a(m_1 + m_2 + m_3) = 0.$$

Cor. From the vertex of the parabola only one real normal can be drawn.

Note. The points on the parabola the normals at which meet in a point are called *co-normal points*. The *co-ordinates* of the co-normal points (or *the feet of the three normals*) are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$, $(am_3^2, -2am_3)$.

8·26. Polar Equation.

To find the polar equation of a parabola with the focus as the pole and the axis as the initial line.



Let (r, θ) be the polar co-ordinates of any point P on the parabola. Draw PN perp. to the axis. Join SP.

Then, $SP=r$, $\angle XSP=\theta$.

$$r=SP=PM=ZN=ZS+SN=2a+r\cos\theta.$$

$\therefore$ $r(1-\cos\theta)=2a$.

$$\therefore \quad r=\frac{2a}{1-\cos\theta} \text{ or } \frac{2a}{r}=1-\cos\theta.$$

Note. If SZ be taken as the positive direction of the initial line and $\angle ZSP=\theta$, then the equation of the curve is

2a

8·26. Illustrative Examples.

Ex. 1. *Every line parallel to the axis of a parabola may be regarded as a diameter.*

Let any line parallel to the axis be

$$y=\lambda. \quad \ldots \quad \ldots \quad (1)$$

Suppose it bisects a system of chords parallel to $y=mx$.

Then its equation will be

$$y=2a/m \quad \ldots \quad \ldots \quad (2)$$

Comparing (1) and (2), we see $\lambda=2a/m$. $\therefore$ $m=2a/\lambda$. Hence the line (1) bisects the system of chords parallel to $y=\frac{2a}{\lambda}x$, and thus it is a diameter.

Ex. 2. *The tangents at the ends of any chord of a parabola, meet on the diameter which bisects the chord.*

Let PQ be any chord and let its equation be

$$y=mx+c \quad \ldots \quad \ldots \quad (1)$$

and let the tangents at P, Q meet at the point $B(x_1, y_1)$. Then PQ is the chord or contact of tangents drawn from B and hence its equation is

$$yy_1=2a(x+x_1). \quad \ldots \quad \ldots \quad (2)$$

Comparing (2), with (1) we have

$$y_1=2a/m.$$

This shows that the locus of (x_1, y_1) *i.e.* of the point of intersection of tangents is $y=2a/m$, which is no other than the diameter bisecting the system of chords parallel to (1).

Ex. 3. *Find the locus of the point of intersection of a pair of tangents to $y^2=4ax$, the angle included between them being always α.*

What happens if $\alpha=\frac{1}{2}\pi$?

The line $y=mx+\dfrac{a}{m}$ is a tangent to $y^2=4ax$ for all values of m. If this line be a tangent from (x_1, y_1), we have

$$y_1=mx_1+\frac{a}{m},$$

or $$m^2x_1-my_1+a=0. \qquad \ldots \qquad \ldots \qquad (1)$$

Let m_1, m_2 be the roots of (1); then the two tangents from (x_1, y_1) are given by

$$y=m_1x+\frac{a}{m_1}, \quad y=m_2x+\frac{a}{m_2}.$$

Since α is the angle between them

$$\tan\alpha=\frac{m_1-m_2}{1+m_1m_2}. \qquad \ldots \qquad \ldots \qquad (2)$$

Now, $$(m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2$$

$$=\frac{y_1^2}{x_1^2}-\frac{4a}{x_1}=\frac{y_1^2-4ax_1}{x_1^2}. \qquad \ldots \qquad (3)$$

$$1+m_1m_2=1+\frac{a}{x_1}=\frac{a+x_1}{x_1}. \qquad \ldots \qquad \ldots \qquad (4)$$

From (2), (3), (4) we get

$$\tan^2\alpha=\frac{y_1^2-4ax_1}{(a+x_1)^2},$$

or $$y_1^2-4ax_1=(a+x_1)^2\tan^2\alpha.$$

$\therefore$ the locus of (x_1, y_1) *i.e.* of the point of intersection of the tangents is

$$y^2-4ax=(a+x)^2\tan^2\alpha.$$

[This curve is a hyperbola; see Chapter X .] When $\alpha=\frac{1}{2}\pi$, we find from (2), $1+m_1m_2=0$, *i.e.* $1+\dfrac{a}{x_1}=0$, *i.e.* $x_1+a=0$.

Hence the locus in this case is $x+a=0$, *i.e.* the directrix.

Examples VIII(B)

1. Find the co-ordinates of the extremity of the diameter which bisects a system of chords parallel to $y=mx$ and show that the tangent at that point is parallel to its ordinates.

2. Show the sum of the ordinates of the extremities of any chord of a parallel system is constant.

3. Prove that the locus of the middle points of a series of chords of a parabola drawn through the vertex is a parabola.

4. Show that the locus of the middle points of the focal chords of a parabola is a parabola with its vertex at the focus of the first.

5. Find the locus of the mid-points of the chords of $y^2 = 4ax$, which pass through a fixed point (h, k). [C. U.]

6. Find the locus of the mid-points of the normal chords of the parabola $y^2 = 4ax$.

7. Find the locus of the middle points of the chords of a parabola which subtend a right angle at the vertex and prove that all these chords pass through a fixed point on the axis of the curve. [C. U. 1937]

8. Find the locus of the middle points of the chords of contact of orthogonal tangents to the parabola $y^2 = 4ax$.

9. Find the middle point of the line $4x - 3y + 4 = 0$ intercepted by the parabola $y^2 = 8x$.

10. Prove that the locus of the middle point of the chord intercepted by the parabola $y^2 = 4ax$ on any tangent to $y^2 + 4ax = 0$ is a parabola.

11. Find the equation of the pair of tangents from $(1, -4)$ to $y^2 = 8x$.

12. Find the length of the chord of contact of tangents drawn from (x_1, y_1) to $y^2 = 4ax$.

13. Find the area of the triangle formed by the two tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and its chord of contact.

14. Show that the point whose co-ordinates are

$$x = 2t^2 + 3, \; y = 4t + 5,$$

t being a parameter, lies on a parabola and find its equation,

15. Find the point of intersection of the normals at the points t_1, t_2.

16. If the normal at the point t_1 meets the parabola again at t_2, then show that

$$t_2 = -t_1 - \frac{2}{t_1}.$$

17. Show that the semi-latus rectum is a harmonic mean between the segments of any focal chord of a parabola.

18. Prove that the normals at the extremities of a focal chord of a parabola intersect at right angles.

19. If the sides of a quadrilateral inscribed in the parabola $y^2 = 4ax$ make angles $\theta_1, \theta_2, \theta_3, \theta_4$ with its axis, show that

$$\cot\theta_1 + \cot\theta_3 = \cot\theta_2 + \cot\theta_4.$$

20. Prove that the distance of the focus from the intersection of two tangents to a parabola is a mean proportional between the focal radii of the points of contact. [*C. U. 1941*]

21. Find the locus of a point the two tangents drawn from which include an angle of

(i) 60°, (ii) 45°, (iii) 30°.

22. Find the locus of the point of intersection of the tangents to a parabola at two variable pionts t_1, t_2 such that

$$\frac{1}{t_1^2} + \frac{1}{t_2^2} = k \text{ (a constant)}.$$

23. Find the locus of the points of intersection of a pair of tangents to a parabola of which

(i) the product of the cosines ;

(ii) the difference of the cotangents

of the angles of slope is constant and equal to k.

24. Show that a circle described on a focal chord of a parabola as diameter touches its directrix.

25. Show that a circle described on a focal radius of a parabola as diameter touches the tangent at the vertex.

26. Find the co-ordinates of the feet of the three normals to the parabola $y^2=8x$ from the point (30, 24).

[*Use the cubic* (3) *of Art. 8·24.*]

27. Find the equations of the three normals to $y^2=8x$ which pass through (18, 12).

28. The normals at three points P, Q, R of $y^2=4ax$ meet at the point (h, k). Prove that the $C.G.$ of the triangle PQR lies on the axis at a distance $\frac{2}{3}(h-2a)$ from the vertex. [*A. U.*]

29. If the normals at points t_1, t_2 meet on the parabola then $t_1t_2=2$.

30. Find the length of the focal chord of the parabola which is inclined at an angle of 60° to the axis.

31. Show that the least focal chord of a parabola is the latus rectum.

32. In the parabola, prove that

(i) the sum of the reciprocals of the segments of any focal chord is constant ;

(ii) the sum of the reciprocals of two perpendicular focal chords is constant.

[*Use polar equation, Art. 8·25.*]

33. Show that the area of the triangle inscribed in a parabola is double that of the triangle formed by the tangents at the vertices.

34. Show that the orthocentre of the triangle formed by three tangents to a parabola lies on the directrix.

CHAPTER IX

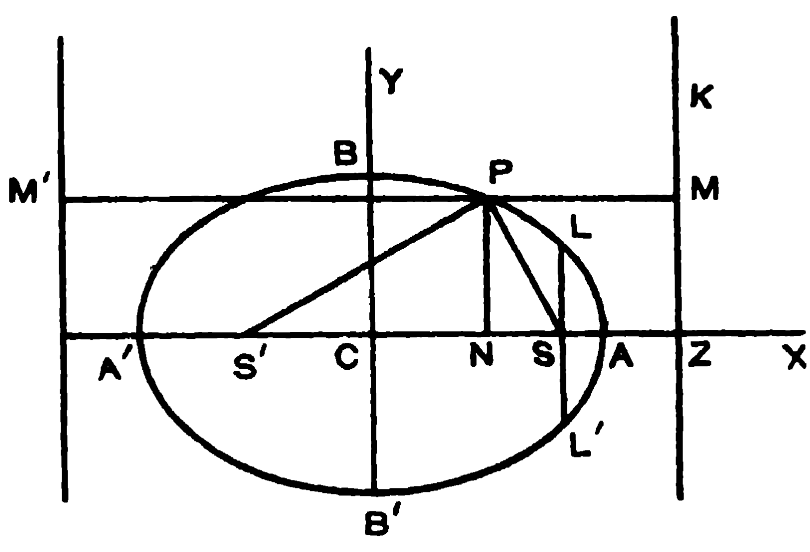
ELLIPSE

9·1. Definitions.

An *Ellipse* is the locus of a point which moves so that its distance from a fixed point bears constant ratio, less than unity, to its distance from a fixed line. The fixed point is called the *Focus*, the fixed line the *Directrix* and the ratio is called the *Eccentricity*, usually denoted by e, (< 1).

9·2. Standard Equation.

To find the equation to an ellipse, given the focus, directrix and eccentricity.



Let S be the focus, ZK the directrix and e (< 1) be the eccentricity. From S draw SZ perp. to ZK and divide SZ *internally* at A and *externally* at A' in the ratio $e : 1$, so that

$$\frac{SA}{AZ} = \frac{SA'}{A'Z} = e.$$

Let C be the mid-point of AA'. Draw CY perp. to AA'.

Take C as origin and CX, CY as x-axis and y-axis respectively. Let $AA' = 2a$; then $CA = CA' = a$.

We have

$$e = \frac{SA}{AZ} = \frac{SA'}{AZ'} = \frac{CA - CS}{CZ - CA}$$

$$= \frac{CS + CA'}{A'C + CZ} = \frac{a - CS}{CZ - a} = \frac{a + CS}{a + CZ}$$

$$= \frac{2CS}{2a} = \frac{2a}{2CZ}$$

$$\therefore \quad CS = ae, \quad \cdots \quad \cdots \quad (1)$$

$$\text{and} \quad CZ = a/e. \quad \cdots \quad \cdots \quad (2)$$

$\therefore$ focus is $(ae, 0)$ and directrix is $x - a/e = 0$.

Let (x, y) be the co-ordinates of any point P on the ellipse. Draw PN perp. to AA' and PM perp. to the directrix. From the definition of ellipse

$$SP = e.PM,$$

$$\text{or} \quad SP^2 = e^2 PM^2.$$

$$\therefore \quad (x - ae)^2 + y^2 = e^2(x - a/e)^2.$$

$$i.e. \quad (1 - e^2)x^2 + y^2 = (1 - e^2)a^2,$$

$$i.e. \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \quad \cdots \quad \cdots \quad (3)$$

Putting $x = 0$ in (3), we find, $y = \pm a\sqrt{1 - e^2}$, which shows that the axis of y meets the ellipse in two points B, B' (which are real, since $e < 1$), lying on opposite sides of C, such that

$$CB = CB' = a\sqrt{1 - e^2}.$$

Let us take $\quad CB = CB' = b$, then

$$\mathbf{b^2 = a^2(1 - e^2).} \quad \cdots \quad (4)$$

Then the equation (3) becomes

$$\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.} \quad \cdots \quad (5)$$

This is the standard equation of the ellipse.

Note 1. The points A, A' are called the **vertices**, AA' the **major axis**, BB' the **minor axis** and C the **centre** of the ellipse.

Note 2. If we take a point S' on the negative side of the origin such that $CS'=CS=ae$ and another point Z' such that $CZ'=CZ=a/e$, and we draw $Z'K'$ perp. to AA', PM' perp. to $Z'K'$, then it can be easily shown from the equation (5) that the relation $S'P^2 = e^2PM'^2$ is satisfied and hence S' is the *second focus* and $Z'K'$ the *second directrix*. Thus *the ellipse has a second focus and a second directrix.*

Note 3. With reference to the above equation of the ellipse, the following results should be noted :

(i) Co-ordinates of **centre : (0, 0).**

(ii) Co-ordinates of **focii : ($\pm$ae, 0).**

(iii) Equation of **Major axis : y = 0.**

(iv) Equation of **Minor axis : x = 0.**

(v) Equations of **Directrices : x = $\pm$a/e.**

(vi) **Length of Major axis = 2a.**

(vii) **Length of Minor axis = 2b.**

(viii) **Eccentricity** : $e^2 = \dfrac{a^2-b^2}{a^2}$.

(ix) **Latus rectum** $= 2\dfrac{b^2}{a}$.

Note 4. From (5), we have

$$\frac{PN^2}{b^2} = \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2-x^2}{a^2} = \frac{(a+x)(a-x)}{a^2} = \frac{AN.A'N}{a^2}.$$

Hence the relation (5) is equivalent to the well-known property of the ellipse

$$PN^2 : AN.A'N = CB^2 : CA^2.$$

Note 5. *Peculiarities of the curve.*

(i) The ellipse is symmetrical about the origin, since by putting $-x$, $-y$ for x, y in the equation of the curve, the form of the equation is not altered ; so *the origin is its centre.*

(ii) *The curve is symmetrical about the x-axis,* (*i.e. the major axis*) since by putting $-y$ for y, leaving x unchanged, the form of the equation remains unchanged. Similarly, *the curve is symmetrical about the y-axis* (*i.e. the minor axis*).

(iii) Since $y^2 = b^2(1 - x^2/a^2)$, if $x^2/a^2 > 1$, *i.e.* if x has any value not lying between $-a$ and a, y^2 is negative and hence y is not real. Similarly x is not real when y has any value not lying between $-b$ and b.

This shows that the curve lies entirely within the rectangle formed by the lines $x = \pm a$, $y = \pm b$. Hence, *the ellipse is called a closed curve.*

9·3. General Equation.

To find the equation of an ellipse whose focus is (α, β), whose directrix is $lx + my + n = 0$ and eccentricity is e (< 1).

Let $P(x, y)$ be any point on the ellipse, S be the focus and PM be the perp. on the directrix.

$$\therefore \quad SP = ePM,$$

$$\text{or} \quad SP^2 = e^2 PM^2.$$

$$\therefore \quad (x - \alpha)^2 + (y - \beta)^2 = e^2 \frac{(lx + my + n)^2}{l^2 + m^2}.$$

Note. This equation on simplification can be put in the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, whence it can be easily shown that $h^2 < ab$. For an illustration see Ex. 1 of Art. 9.

Henceforth standard equation will be used unless otherwise stated.

Special form.

By changing the origin to (α, β), axes remaining parallel, the equation

$$\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1 \qquad \ldots \qquad (1)$$

reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \ldots \qquad (2)$$

Hence an equation of the form (1) always represents an ellipse whose centre is (α, β) and axes are of lengths $2a$, $2b$ and equations of whose axes are $x = \alpha$, $y = \beta$.

The equation (1) when simplified assumes the form

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0. \quad \cdots \quad (3)$$

So we conclude that *an equation of form* (3) (where A and B are both positive and unequal), *always represents an ellipse with its principal axes parallel to the co-ordinate axes* and *is* a *real ellipse*, a *point ellipse* or an *imaginary ellipse* according as the expression $G^2/A + F^2/B - C > = \text{or} < 0$.

9·4. Latus Rectum.

The *Latus rectum* of the ellipse is the double ordinate through either focus. If LSL' is a latus rectum, $SL = SL'$.

Let the co-ordinates of L be (ae, l), so that $SL = l$.

$$\frac{a^2e^2}{a^2} + \frac{l^2}{b^2} = 1, \quad \text{or} \quad \frac{l^2}{b^2} = 1 - \qquad b^2$$

$$\therefore \qquad l^2 = \frac{b^4}{a^2}. \quad \therefore \quad l = \frac{b^2}{a}.$$

$$\therefore \qquad \textit{Latus rectum} = 2\frac{\mathbf{b}^2}{\mathbf{a}}$$

9·5. Focal Distances of a point.

Let (x_1, y_1) be the co-ordinates of P. From fig. of Art. 9·2,

$$\mathbf{SP} = ePM = eNZ = eCZ - eCN = \mathbf{a - ex_1}$$

$$\mathbf{S'P'} = ePM' = eNZ' = eCZ' + eCN = \mathbf{a + ex_1}.$$

$$\therefore \qquad \mathbf{SP + S'P} = a - ex_1 + a + ex_1 = \mathbf{2a}.$$

Thus, *the sum of the focal distances of any point on the ellipse is constant and equal to the major axis.*

9·6. Position of a point in relation to an ellipse.

The point (x_1, y_1) *lies outside, upon or inside the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *according as* $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > = \text{ or } < 0$.

Let Q be the point (x_1, y_1). Draw QN perp. to AA' cutting the ellipse in P. Then the abscissa of P is x_1 and let its ordinate be y', so that P is (x_1, y').

$$\therefore \qquad \frac{x_1^2}{a^2} + \frac{y'^2}{b^2} = 1, \quad \textit{i.e.} \quad \frac{y'^2}{b^2} = 1 - \frac{x_1^2}{a^2}.$$

Now, Q lies outside, upon or inside the ellipse

according as $QN^2 >=$ or $< PN^2$

" $y_1^2 >=$ or $< y'^2$

" $\frac{y_1^2}{b^2} >=$ or $< \frac{y'^2}{b^2}$

" $\frac{y_1^2}{b^2} >=$ or $< 1 - \frac{x_1^2}{a^2}$

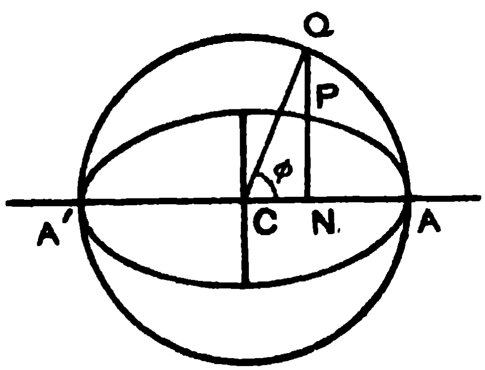
" $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 >=$ or < 0.

9·7. Auxiliary Circle.

The circle discribed upon the major axis of the ellipse as diameter, is called the *Auxiliary circle* of the ellipse.

The equation of the auxiliary circle *w. r. t.* the axes of the ellipse as axes of co-ordinates is

$$x^2 + y^2 = a^2.$$



Let P be any point on the ellipse. Draw PN prep. to the major axis and produce it to meet the auxiliary circle in Q. Since co-ordinates of P are (CN, PN), co-ordinates of Q are (CN, QN); we have from the equation of the ellipse and the auxiliary circle,

$$\frac{CN^2}{a^2} + \frac{PN^2}{b^2} = 1, \quad \frac{CN^2}{a^2} + \frac{QN^2}{a^2} = 1.$$

$$\frac{PN^2}{b^2} = \frac{QN^2}{a^2}$$

or $$\frac{PN}{QN} = \frac{b}{a}.$$

Points P and Q are called *corresponding* points. Thus, the ratio of the ordinates of a pair of corresponding points is constant.

9·8. Eccentric Angle and Parametric representation.

In the fig. of Art. 9·7, join CQ. Then $\angle QCA$ is called eccentric angle and is here denoted by φ.

Thus, *the Eccentric angle of any point on an ellipse is the angle which the line joining the centre to the coorresponding point on the auxiliary circle makes with the major axis.*

From the right-angled $\triangle QCN$,

$$CN = CQ \cos\varphi = a\cos\varphi.$$

If (x, y) be the co-ordinates of P, we have $x = a\cos\varphi$.

$$\therefore \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = 1 - \frac{a^2\cos^2\varphi}{a^2} = \sin^2\varphi.$$

$$\therefore \quad y = b\sin\varphi.$$

Thus *co-ordinates of P* are $(a\cos\varphi, b\sin\varphi)$,

and *co-ordinates of Q* are $(a\cos\varphi, a\sin\varphi)$.

Since $\quad \mathbf{x = a\cos\varphi}$

$\qquad\quad \mathbf{y = b\sin\varphi}$

satisfy the equation of the ellipse whatever be the value of φ, these two together give the *parametric representation* of the ellipse.

The point whose co-ordinates are $(a\cos\varphi, b\sin\varphi)$ is, for sake of brevity, very often called the *point φ*.

9·9. Intersection of a line and the ellipse.

To find the points of intersection of the line

$$y = mx + c \qquad \dots \qquad (1)$$

with the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1. \qquad \dots \qquad \dots \qquad (2)$

The abscissæ of the points of intersection of (1) and (2) are given by the quadratic in x, obtained by eliminating y between (1) and (2), *i.e.* by

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1,$$

or $\quad (a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0. \quad \cdots \quad (3)$

This being a quadratic in x has two roots (say, x_1, x_2) real, coincident or imaginary.

Hence *every line meets the ellipse in two points real, coincident* or *imaginary.*

The points of intersection will be real, coincident or imaginary, according as the roots of (3) are real, coincident or imaginary *i.e.* according as

$$4a^4m^2c^2 - 4(a^2m^2 + b^2) \times a^2(c^2 - b^2) > = \text{ or } < 0$$

i.e. according as $a^2m^2 > = \text{ or } < c^2 - b^2$

i.e. ,, $c^2 < = > a^2m^2 + b^2$.

The ordinates of the points of intersection, y_1, y_2 can be obtained by substituting x_1, x_2 for x in (1).

Thus $y_1 = mx_1 + c,\ \ y_2 = mx_2 + c.$ ... (4)

9·10. Tangent at a point.

To find the equation of the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots \qquad (1)$$

at the point (x_1, y_1) *on it.*

Let P be the given point (x_1, y_1) and $Q(x_2, y_2)$ be a point on the ellipse very close to P. The equation to PQ is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \qquad \cdots \qquad (2)$$

Since P, Q lie on the ellipse (1).

$$\therefore \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \text{ and } \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1.$$

$\therefore$ by subtraction

$$\frac{x_1^2 - x_2^2}{a^2} + \frac{y_1^2 - y_2^2}{b^2} = 0,$$

$$i.e. \quad \frac{(x_1 + x_2)(x_1 - x_2)}{a^2} = -\frac{(y_1 + y_2)(y_1 - y_2)}{b^2}.$$

$$\frac{y_1 - y_2}{x_1 - x_2} = -\frac{b^2}{a^2}\frac{x_1 + x_2}{y_1 + y_2}. \qquad (3)$$

$\therefore$ by (3), the equation (2) of the secant PQ becomes

$$y - y_1 = -\frac{b^2}{a^2}\frac{x_1 + x_2}{y_1 + y_2}(x - x_1). \quad \cdots \quad (4)$$

When Q moves and ultimately coincides with P, the secant PQ becomes the tangent at P ; in this case $y_2 = y_1$. Hence putting $y_2 = y_1$, the equation of the tangent at P becomes

$$y - y_1 = -\frac{b^2}{a^2}\frac{x_1}{y_1}(x - x_1),$$

i.e. $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

Hence the *equation of the tangent at* (x_1, y_1) is

$$\mathbf{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.}$$

9·11. Condition of tangency of a line.

To find the condition that the line $y = mx + c \quad \cdots \quad (1)$ shall touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \cdots \quad \cdots \quad (2)$$

The abscissæ of the points of intersection of (1) and (2) are given by

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1,$$

i.e. $(a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0. \quad \ldots \quad (3)$

The line (1) will touch the ellipse if the two points of intersection are coincident *i.e.* if the roots of (3) are equal

i.e. if $4a^4m^2c^2 = 4(a^2m^2 + b^2)a^2(c^2 - b^2)$

i.e. if $c^2 = a^2m^2 + b^2$

i.e. if $\mathbf{c = \pm \sqrt{a^2m^2 + b^2}}. \ldots \quad \ldots \quad (4)$

Hence each of the lines $\mathbf{y = mx \pm \sqrt{a^2m^2 + b^2}}$ is always a tangent to the ellipse (2), whatever be the value of *m*.

Point of contact.

Let (x_1, y_1) be the point of contact of the line.

Then $\quad x_1 =$ equal root of (3)

$$= -\frac{a^2mc}{a^2m^2+b^2} = \mp\frac{a^2m\sqrt{a^2m^2+b^2}}{a^2m^2+b^2} \text{ by (4)}$$

$$= \mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}.$$

$$y_1 = mx_1 + c = \mp\frac{a^2m^2}{\sqrt{a^2m^2+b^2}} \pm \sqrt{a^2m^2+b^2}$$

$$= \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}.$$

$\therefore$ *co-ordinates of the point of contact* are

$$\left(\mp\frac{a^2m}{\sqrt{a^2m^2+b^2}},\ \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right). \quad \dots \quad (5)$$

Note. Similarly it can be shown that $x\cos\alpha + y\sin\alpha = p$ will touch the ellipse if

$$\mathbf{p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha},$$

the point of contact being $(a^2\cos\alpha/p,\ b^2\sin\alpha/p)$ and $lx+my+n=0$ will touch the ellipse if $\quad \mathbf{a^2l^2+b^2m^2=n^2}$

the point of contact being $(-a^2l/n,\ -b^2m/n)$.

Alternative Method.

Let (x_1, y_1) be the point where the line (1) touches the ellipse. The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \quad \dots \quad \dots \quad (6)$$

(1) can be written as $mx - y = -c$. $\quad \dots \quad \dots \quad (7)$

$\therefore$ (6) and (7) must represent the same line.

$$\therefore \quad \frac{x_1}{a^2m} = \frac{y_1}{-b^2} = \frac{1}{-c}.$$

$$\therefore \quad x_1 = -\frac{a^2m}{c}, \quad y_1 = \frac{b^2}{c}. \quad \dots \quad (8)$$

$$\therefore \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1. \qquad \therefore \quad \frac{a^4m^2}{c^2} + \frac{b^4}{b^2c^2} = 1.$$

$$\therefore \quad c^2 = a^2m^2 + b^2, \quad \textit{i.e.} \quad c = \pm\sqrt{a^2m^2+b^2}. \quad \dots \quad (9)$$

Substituting the value of c from (9) in (8), co-ordinates of the point of contact will be obtained.

9·12. Equation of a chord in parametric co-ordinates.

To find the equation of the line joining the two points $(a \cos \theta, b \sin \theta)$ *and* $(a \cos \varphi, b \sin \varphi)$.

The equation of the line joining them is

$$\frac{x - a \cos \theta}{a(\cos \varphi - \cos \theta)} = \frac{y - b \sin \theta}{b(\sin \varphi - \sin \theta)},$$

or $$\frac{x - a \cos \theta}{2a \sin \frac{1}{2}(\theta+\varphi) \sin \frac{1}{2}(\theta-\varphi)} = -\frac{y - b \sin \theta}{2b \cos \frac{1}{2}(\theta+\varphi) \sin \frac{1}{2}(\theta-\varphi)},$$

or $$\frac{x - a \cos \theta}{a \sin \frac{1}{2}(\theta+\varphi)} = -\frac{y - b \sin \theta}{b \cos \frac{1}{2}(\theta+\varphi)},$$

or $$\frac{x}{a} \cos \frac{1}{2}(\theta+\varphi) + \frac{y}{b} \sin \frac{1}{2}(\theta+\varphi)$$

$$= \cos \theta \cos \frac{1}{2}(\theta+\varphi) + \sin \theta \sin \frac{1}{2}(\theta+\varphi)$$

$$= \cos\{\theta - \cos \frac{1}{2}(\theta+\varphi)\}$$

$$= \cos \frac{1}{2}(\theta-\varphi).$$

$\therefore$ the required equation is

$$\frac{x}{a} \cos \frac{1}{2}(\theta+\varphi) + \frac{y}{b} \sin \frac{1}{2}(\theta+\varphi) = \cos \frac{1}{2}(\theta-\varphi).$$

Cor. Putting $\theta = \varphi$, the *equation of the tangent at* φ is

$$\frac{\mathbf{x}}{\mathbf{a}} \cos \varphi + \frac{\mathbf{y}}{\mathbf{b}} \sin \varphi = 1.$$

9·13. Normal at a point.

(A) *To find the equation of the normal of the ellipse at* (x_1, y_1).

The tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$,

$$\text{i.e.} \quad y = -\frac{b^2}{a^2} \frac{x_1}{y_1} x + \frac{b^2}{y_1}. \qquad \cdots \quad (1)$$

Any line through (x_1, y_1) is

$$y - y_1 = m(x - x_1). \qquad \cdots \quad (2)$$

If (2) be normal at (x_1, y_1), it must be perp. to (1).

$$\therefore \quad m \times \left(-\frac{b^2 x_1}{a^2 y_1}\right) = -1, \text{ i.e. } m = \frac{a^2 y_1}{b^2 x_1}.$$

The equation of the normal is therefore

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1),$$

$$\text{i.e. } \frac{\mathbf{x - x_1}}{\frac{\mathbf{x_1}}{\mathbf{a^2}}} = \frac{\mathbf{y - y_1}}{\frac{\mathbf{y_1}}{\mathbf{b^2}}}. \quad \cdots \quad (3)$$

(B) *To find the equation of the normal at the point φ.*

The co-ordinates of the point are $(a \cos \varphi, b \sin \varphi)$ Putting $x_1 = a \cos \varphi, y_1 = b \sin \varphi$ in the above equation (3) it becomes

$$\frac{a(x - a \cos \varphi)}{\cos \varphi} = \frac{b(y - b \sin \varphi)}{\sin \varphi},$$

$$\text{or} \quad ax \sec \varphi - a^2 = by \operatorname{cosec} \varphi - b^2.$$

The required normal is therefore

$$\mathbf{ax \sec \varphi - by \operatorname{cosec} \varphi = a^2 - b^2.}$$

9·14. Number of tangents from a point.

To show that from any point there can be drawn two tangents to an ellipse.

The line is $\quad y = mx + \sqrt{a^2m^2 + b^2} \quad \ldots \quad (1)$

is a tangent to the ellipse for all values of m.

The problem of drawing a tangent from (x_1, y_1) reduces to that of choosing the parameter m so that (1) may pass through (x_1, y_1), the condition for which is

$$y_1 - mx_1 = \sqrt{a^2m^2 + b^2},$$

$$\text{or} \quad (y_1 - mx_1)^2 = a^2m^2 + b^2,$$

$$\text{or} \quad (x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 - b^2) = 0. \quad \cdots \quad (2)$$

This equation being a quadratic in m, gives the m's of the two tangents drawn from (x_1, y_1). Hence there are two tangents from (x_1, y_1).

Evidently *the two tangents from* (x_1, y_1), *will be real, coincident or imaginary* according as the roots of (2) are real, coincident or imaginary ; that is according as

$$4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - b^2) > = \text{or} < 0$$

i.e. according as $$b^2x_1^2 + a^2y_1^2 - a^2b^2 > = \text{or} < 0$$

i.e. ,, $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > = \text{or} < 0$$

i.e. according as the point (x_1, y_1) *is outside, upon or inside the ellipse.*

9·15. Pole and Polar.

The Pole and Polar with respect to an ellipse are defined exactly in the same way as in the case of a circle. [Art. 6·12]

(A) *To find the polar of the point* (x_1, y_1) *w. r. t. the ellipse* $x^2/a^2 + y^2/b^2 = 1$.

If the points of contact be (x_2, y_2), (x_3, y_3) the tangents at these points are

$$xx_2/a^2 + yy_2/b^2 = 1$$
$$xx_3/a^2 + yy_3/b^2 = 1.$$

Since both these tangents pass through $P(x_1, y_1)$.

$$\therefore \quad x_1x_2/a^2 + y_1y_2/b^2 = 1$$
$$x_1x_3/a^2 + y_1y_3/b^2 = 1.$$

These relations show that the two points A, B are situated on the line

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1,$$

which is therefore the reqd. equation of the polar of the point.

Cor. *The polar of the focus is the corresponding directrix.*

(B) *To find the pole of the line* $lx + my + n = 0$ *w. r. . the above ellipse.*

Let (x_1, y_1) be the reqd. pole.

The polar of (x_1, y_1) *w. r. t.* the ellipse is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0.$$

This must be identical with

$$lx+my+n=0.$$

$$\frac{x_1}{a^2 l}=\frac{y_1}{b^2 m}=-\frac{1}{n}.$$

$\therefore$ reqd. pole is $\left(-\frac{l}{n}a^2,\quad -\frac{m}{n}b^2\right)$.

9·16. Properties of Pole and Polar.

(i) *If the polar of a point P w. r. t. an ellipse passes through Q, the polar of Q passes throuyh P.*

(ii) *The point of intersection of any two lines is the pole of the line joining the poles of the lines.*

(iii) *If the pole of the line $lx+my+n=0$ w. r. t. an ellipse lies on the line $l'x+m'y+n'=0$, then the pole of the line $l'x+m'y+n'=0$ lies on $lx+my+n=0$.*

The proofs of these three are exactly similar to those given in the case of a circle. (*Art. 6·13*).

9·17. Chord of contact.

The chord of contact of tangents drawn from (x_1, y_1) to the ellipse is, as we have seen in the case of a circle and a parabola, nothing but the polar of the point and hence its equation is

$$\frac{\mathbf{xx_1}}{\mathbf{a^2}}+\frac{\mathbf{yy_1}}{\mathbf{b^2}}=\mathbf{1}.$$

9·18. Director circle.

To show that the locus of the point of intersection of a pair of perpendicular tangents to an ellipse is a circle.

Any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2}, \quad \cdots \quad (1)$$

and hence a perpendicular tangent is

$$y = -\frac{1}{m}x + \sqrt{\frac{a^2}{m^2} + b^2}. \quad \cdots \quad (2)$$

Let (x_1, y_1) be their point of intersection.

$$\therefore \quad y_1 - mx_1 = \sqrt{a^2m^2 + b^2} \quad \cdots \quad (3)$$

$$my_1 + x_1 = \sqrt{a^2 + b^2m^2}. \quad \cdots \quad (4)$$

Squaring and adding (3) and (4), we get

$$(1 + m^2)(x_1^2 + y_1^2) = (1 + m^2)(a^2 + b^2),$$

$$i.e. \quad x_1^2 + y_1^2 = a^2 + b^2.$$

$\therefore$ the locus of (x_1, y_1) is the circle

$$\mathbf{x^2 + y^2 = a^2 + b^2}.$$

The centre of this circle is the centre of the ellipse and radius is equal to the length of the line joining the ends of major and minor axes.

Def. The circle which is the locus of the point of intersection of a pair of perpendicular tangents is called *Director circle.*

Alternative Method.

Any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2} \quad \cdots \quad \cdots \quad (1)$$

where m is a variable parameter.

If it passes through (x_1, y_1) then

$$y_1 = mx_1 + \sqrt{a^2m^2 + b^2},$$

$$i.e. \ (y_1 - mx_1)^2 = a^2m^2 + b^2$$

$$\text{or} \quad (x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 - b^2 = 0 \quad \cdots \quad (2)$$

This quadratic gives the m's of the two tangents through (x_1, y_1). If m_1, m_2 be the roots of (2), the two tangents will be perpendicular

$$\text{if} \quad m_1m_2 = -1,$$

$$i.e. \text{ if } \frac{y_1^2 - b^2}{x_1^2 - a^2} = -1,$$

$$i.e. \quad x_1^2 + y_1^2 = a^2 + b^2.$$

$\therefore$ the locus of (x_1, y_1) is the circle $x^2 + y^2 = a^2 + b^2$

9·19. Illustrative Examples.

Ex. 1. *Find the equation of the ellipse whose focus is* (2, 3) *eccentricity* $\frac{1}{\sqrt{13}}$ *and directrix* $x-y+13=0$. [C. U.]

Let (x, y) be any point on the ellipse. Then from the definition of ellipse, we have from Art. 9·3,

$$SP^2=e^2PM^2.$$

$$\therefore \quad (x-2)^2+(y-3)^2=\frac{1}{13}\cdot\left(\frac{x-y+13}{\sqrt{2}}\right)^2$$

which on simplification reduces to

$$25x^2+2xy+25y^2-130x-130y+169=0.$$

This is the reqd. equation of the ellipse.

Ex. 2. *The locus of a point which moves such that the sum of its distances from two fixed points is constant is an ellipse.*

Take the line joining the given points S, S' as the x-axis and its middle point O as origin. Let the co-ordinates of S, S' be (c, o), $(-c, o)$. If (x, y) be a point P on the curve, we have

$$SP+S'P=\text{constant}=2a \text{ say}.$$

$$\therefore \quad \sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a.$$

$$\therefore \quad \sqrt{(x-c)^2+y^2}-2a=-\sqrt{(x+c)^2+y^2}.$$

On squaring, we obtain

$$(x-c)^2+y^2-4a\sqrt{(x-c)^2+y^2}+4a^2=(x+c)^2+y^2.$$

Simplifying, rearranging and dividing by 4, we get

$$a\sqrt{(x-c)^2+y^2}=a^2-cx.$$

Squaring again,

$$a^2(x-c)^2+a^2y^2=a^4-2a^2cx+c^2x^2,$$

$$i.e. \quad x^2(a^2-c^2)+y^2a^2=a^4-a^2c^2=a^2(a^2-c^2).$$

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1.$$

Since $SP+S'P > SS'$ $\therefore$ $a > c$ $\therefore$ a^2-e^2 is positive and say equal to b^2, where b^2 obviously less than a^2.

Hence $$\frac{x^2}{a^2}+\frac{y}{b^2}=1.$$

The locus of P is thus an ellipse.

Ex. 3. *Show that a circle is a limiting case of an ellipse.*

When $b=a$, the ellipse $x^2/a^2+y^2/b^2=1$ becomes the circle $x^2+y^2=a^2$, and $e^2=\dfrac{a^2-b^2}{a^2}=0$. Also in such a case, $CA=e\,CZ$ or $CA/CZ=e=0$; since CA is finite, CZ must be infinitely large.

Again, $CS'=CS=e.CA=0.CA=0$, the two focii conicide.

Thus *a cirlce may be regarded as a limiting case of an ellipse in which the major axis is equal to minor axis, eccentricity is equal to zero, the two focii are coincident with the centre aud the directrices are situated at infinity.*

Ex. 4. *Find the length of the chord intercepted on $y=mx+c$ by the ellipse.*

Let (x_1, y_1), (x_2, y_2) be the points of intersection of the line and the ellipse, and let D be the reqd. length. Then x_1, x_2 are the roots of the quadratic.

$$(a^2m^2+b^2)x^2+2a^2mcx+a^2(c^2-b^2)=0. \qquad \dots \quad \dots \quad (1)$$

[*See Art. 9·3*]

$$\therefore \quad (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$
$$=\frac{4a^4m^2c^2}{(a^2m^2+b^2)^2}-\frac{4a^2(c^2-b^2)}{a^2m^2+b^2}$$
$$=\frac{4a^2}{(a^2m^2+b^2)^2}\{a^2m^2c^2+(b^2-c^2)(a^2m^2+b^2)\}$$
$$=\frac{4a^2b^2}{(a^2m^2+b^2)^2}(a^2m^2+b^2-c^2). \qquad \dots \quad \dots \quad (2)$$

Since $y_1=mx_1+c$, $y_2=mx_2+c$. $\therefore$ $y_1-y_2=m(x_1-x_2)$. ... (3)

$$D^2=(x_1-x_2)^2+(y_1-y_2)^2=(x_1-x_2)^2(1+m^2) \text{ by (3)}.$$

$\therefore$ by (2) and (3),

$$D^2=\frac{4a^2b^2(1+m^2)}{(a^2m^2+b^2)^2}(a^2m^2+b^2-c^2).$$

Hence D can be obtained.

Ex. 5. *Find the point of intersection of the tangents at θ and ϕ to the ellipse.*

Let (x_1, y_1) be the point of intersection. The polar of (x_1, y_1) is

$$xx_1/a^2+yy_1/b^2=1. \qquad \dots \quad \dots \quad (1)$$

The line joining θ, ϕ, is

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\phi)+\frac{y}{b}\sin\tfrac{1}{2}(\theta+\phi)=\cos\tfrac{1}{2}(\theta-\phi). \quad (2)$$

(1) and (2) must be identical

$$\therefore \quad \frac{x_1/a^2}{\frac{1}{a}\cos\frac{1}{2}(\theta+\phi)}=\frac{y_1/b^2}{\frac{1}{b}\sin\frac{1}{2}(\theta+\phi)}=\frac{1}{\cos\frac{1}{2}(\theta-\phi)}$$

whence we get $x_1=a\dfrac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)},\ y_1=b\dfrac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}$.

Otherwise:

The tangents at θ and ϕ are

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1=0,$$

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1=0.$$

Solving for x, y by the rule of Cross-Multiplication,

$$\frac{x/a}{\sin\theta-\sin\phi}=\frac{y/b}{\cos\phi-\cos\theta}=\frac{-1}{\sin\phi\cos\theta-\cos\phi\sin\theta}=\frac{1}{\sin(\theta-\phi)},$$

i.e. $$\frac{x}{2a\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}=\frac{y}{2b\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}$$

$$=\frac{1}{2\sin\frac{1}{2}(\theta-\phi)\cos\frac{1}{2}(\theta-\phi)}.$$

On simplification we get the values of x, y, the co-ordinates of the point of intersection.

Ex. 6. *Find the locus of the foot of the perpendicular drawn upon any tangent to the ellipse $x^2/a^2+y^2/b^2=1$*

(i) *from the centre,*

(ii) *from either focus.*

(i) Let $x\cos\alpha+y\sin\alpha=p$ be a tangent to the ellipse ; then

$$p^2=a^2\cos^2\alpha+b^2\sin^2\alpha.$$

Also, (p, α) are the polar co-ordinates of the foot of the perpendicular.

$\therefore$ the polar equation of the locus of the foot is

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta.$$

Multiplying both sides of this by r^2,

$$r^4 = a^2r^2\cos^2\theta + b^2r^2\sin^2\theta,$$

or $$(x^2+y^2)^2 = a^2x^2+b^2y^2.$$

This is the Cartesian equation of the locus of the foot *i.e.* of the *pedal of the ellipse w. r. t. centre.*

(ii) The line $y = mx + \sqrt{a^2m^2+b^2}$ ... (1) is a tangent to the ellipse for all values of m.

The line through the focus (ae, o) perp. to (1)

$$\text{is} \quad y = -\frac{1}{m}(x - ae). \qquad (2)$$

Now (1) and (2) can be written as

$$y - mx = \sqrt{a^2m^2+b^2},$$
$$my + x = ae = \sqrt{a^2-b^2}.$$

Eliminating m between (1) and (2), we get the locus of the point of intersection of the lines (1) and (2) *i.e.* of the foot of the perpendicular.

Now (1) and (2) can be written as

$$y - mx = \sqrt{a^2m^2+b^2}. \quad \ldots \quad (3)$$
$$my + x = \sqrt{a^2-b^2}. \quad \ldots \quad \ldots \quad (4)$$

Now, squaring and adding (3), (4), we get

$$(1+m^2)(x^2+y^2) = a^2(1+m^2),$$

or $$x^2+y^2 = a^2.$$

This is the reqd. equation of the locus, *i.e.* of the **pedal of the ellipse w. r. t. either focus.** This is the *Auxiliary circle.*

Examples IX(A)

[*Unless otherwise stated, by an ellipse is meant here an ellipse represented by its standard equation.*]

1. Find the equation to the ellipse

(i) whose focus is $(-2, 3)$, eccentricity is $\sqrt{\frac{1}{3}}$ and directrix is $x - y + 7 = 0$;

(ii) whose focus is $(-1, 1)$, eccentricity is $\frac{1}{2}$ and directrix is $x - y + 3 = 0$. [*C. U. 1945*]

2. Find the centre of the ellipse whose eccentricity is $\frac{1}{2}$, focus S is $(0, 0)$, point of intersection of the axis and directrix Z is $(-1, -1)$.

[*ZS is divided internally and externally by A, A' in the ratio of* 2 : 1.]

3. Find the equation of the ellipse (referred to its centre as origin and major axis as x-axis) whose latus rectum is 5 and eccentricity is $\frac{2}{3}$. [*C. U. 1940*]

4. (i) Find the vertices of the ellipse

$$\frac{(x-1)^2}{9}+\frac{(y-3)^2}{5}=1.$$

(ii) Find the eccentricity and latus-rectum of the ellipse

$$3x^2+4y^2+6x-8y=5.$$

5. (i) Verify that the four foci of the two ellipses

$$\frac{x^2}{169}+\frac{y^2}{25}=1,\quad \frac{x^2}{256}+\frac{y^2}{400}=1,$$

are *concyclic.*

(ii) Show that the following ellipses have the same foci

$$\frac{x^2}{25}+\frac{y^2}{6}=1,\quad \frac{x^2}{28}+\frac{y^2}{9}=1,\quad \frac{x^2}{30}+\frac{y^2}{11}=1.$$

6. If the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1,$$

have the same eccentricity, then $a/\alpha=b/\beta$.

7. Find the positions of the following points

$$(4, 1\tfrac{1}{5}), (2, -1), (5, 3)$$

w. r. t. the ellipse $9x^2+25y^2=225$.

8. (i) The distance of a point on the ellipse

$$x^2/6+y^2/2=1$$

from the centre is 2. Find the eccentric angle of the point.

(ii) Find the eccentric angles of the ends of latera recta of $2x^2 + 4y^2 = 1$.

9. Find the length of the chord intercepted by the ellipse $9x^2 + 16y^2 = 144$ on the line $3x + 4y = 12$.

10. Show that the line joining the upper end of one latus rectum and the lower end of the other of an ellipse, passes through its centre.

11. Find the locus of a point which moves such that the sum of its distances from the two points $(\pm 3, 0)$ is 10.

12. (i) Show that the point

$$x = a.\frac{1 - t^2}{1 + t^2}, \quad y = b.\frac{2t}{1 + t^2},$$

where t is a variable parameter, lies on an ellipse.

(ii) Show that the locus of the point of intersection of the lines $\frac{\lambda x}{a} + \frac{y}{b} - \lambda = 0$, $\frac{x}{a} - \lambda \frac{y}{b} + 1 = 0$ where λ is variable, is an ellipse.

13. Prove that the equation of the chord joining the points (x_1, y_1), (x_2, y_2) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

is $$\frac{(x - x_1)(x - x_2)}{a^2} + \frac{(y - y_1)(y - y^2)}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$

and hence deduce the equation of the tangent at (x_1, y_1).

14. (i) Find the equations of the tangents to the ellipse $9x^2 + 16y^2 = 144$, which make equal angles with its axes.

(ii) Find a point on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ such that the tangent at it is equally inclined to the axes.

15. (i) Find the points on the ellipse $x^2 + 4y^2 = 2$ the tangents at which are perpendicular to the line $2x + y - 7 = 0$.

(ii) Find the points on the ellipse $3x^2 + y^2 = 37$ the normal at which is parallel to $5x - 6y + 3 = 0$.

16. Show that the tangent at any point on the ellipse and the tangent at the corresponding point on the auxiliary circle meet on the directrix.

17. If θ, φ be the eccentric angles of a focal chord of an ellipse of eccentricity e, prove that

(i) $\pm e \cos \frac{1}{2}(\theta+\varphi) = \cos \frac{1}{2}(\theta-\varphi)$;

(ii) $\tan \frac{1}{2}\theta \tan \frac{1}{2}\varphi + \dfrac{1 \mp e}{1 \pm e} = 0.$ [*C. U. 1945*]

18. For a system of parallel chords, prove that the sum of the eccentric angles of the extremities of any chord is constant. [*C.U.*]

19. The eccentric angles of two points on an ellipse with $2a$ as the length of its major axis are θ and φ and their join intersects the major axis at a distance c from the centre. Prove that

$$\tan \tfrac{1}{2}\theta \tan \tfrac{1}{2}\varphi = \frac{c-a}{c+a}.$$ [*C. U. 1941*]

20. If PQ, PR are focal chords of an ellipse and 2α, 2β, 2γ are the eccentric angles of P, Q, R ; prove that

$$\tan \alpha \tan \beta \tan \gamma = \cot \alpha.$$

21. Show that the line $lx+my=n$ is a normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ if $\dfrac{a^2}{l^2}+\dfrac{b^2}{m^2}=\dfrac{(a^2-b^2)^2}{n^2}$. [*C. U. 1946*]

22. Show that the condition that normals at (x_1, y_1), (x_2, y_2), (x_3, y_3) on the ellipse $x^2/a^2+y^2/b^2=1$ may be concurrent is

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0.$$ [*C. U.*]

23. If the normal at any point P on the ellipse meet the major and minor axes in G and g; show that

$$PG : Pg = b^2 : a^2.$$

24. Show that the normal and the tangent at any point of an ellipse bisect internally and externally the angle between the focal distances of the point.

[*Suppose the normal meets the axis at* G; *show that* $SG : S'G = SP : S'P$.]

25. Show that the eccentricity of the ellipse in which the normal at one end of a latus rectum passes through an end of the minor axis, is given by the equation

$$e^4 + e^2 - 1 = 0.$$

26. Show that the equation of the normal at (x', y') to the ellipse can be written in the *distance form* as

$$\frac{x - x'}{px'/a^2} = \frac{y - y'}{py'/b^2} = r$$

where p is the perp. from the centre upon the tangent at that point.

27. Find the pole *w. r. t.* the ellipse $x^2/a^2 + y^2/b^2 = 1$ of the following lines

(i) $lx + my = 1$, (ii) $x \cos a + y \sin a - p = 0$;

(iii) the chord joining θ and φ.

28. If the pole of the normal at P lie on the normal at Q, then show that the pole of the normal at Q lies on the normal at P. [*C. U. 1940*]

29. Show that the locus of the poles of a system of parallel chords of an ellipse is a line passing through the centre.

30. (i) Find the locus of the pole of a tangent to the ellipse *w. r. t.* the auxiliary circle.

(ii) Find the locus of the pole of a tangent to the auxiliary circle *w. r. t.* the ellipse.

31. If three points are *collinear*, prove that their polars *w. r. t.* an ellipse are *concurrent.*

32. Prove that the tangents at the ends of a focal chord of an ellipse meet on the directrix.

33. (i) Show that the equation of the chord of the ellipse joining two points whose eccentric angles are $(\alpha \pm \beta)$ is

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = \cos\beta.$$

(ii) Prove that the tangents to the ellipse at points whose eccentric angles are $(\alpha \pm \beta)$ meet at the point

$$(a\cos\alpha\sec\beta,\ b\sin\alpha\sec\beta).$$

34. Find the locus of the point of intersection of tangents at two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$ when

(i) the sum of their eccentric angles is constant $= (2\delta)$,

(ii) the difference of their eccentric angles is constant $= (2\delta)$.

35. Show that the line joining the extremities of any two diameters of an ellipse which are at right angles to one another will always touch a fixed circle.

36. Prove that the locus of poles of such chords of an ellipse as touch a concentric and co-axial ellipse is another concentric and co-axial ellipse. [*C. U. 1928*]

37. Find the locus of a point such that its polar *w. r. t.* the ellipse $b^2x^2 + a^2y = a^2b^2$ passes through a fixed point (h, k).

38. Find the locus of the poles of normal chords of the ellipse.

39. Prove that the locus of the pole of a chord of an ellipse

(i) which subtends a right angle at the centre,

(ii) which touches the director circle,

is in both cases, a concentric ellipse.

40. A point is such that the perpendicular from the centre on its polar *w. r. t.* the ellipse $x^2/a^2 + y^2/b^2 = 1$ is constant and equal to c; show that its locus is the ellipse $x^2/a^4 + y^2/b^4 = 1/c^2$. [*P. U. 1945*]

41. Find the director circle of $4x^2 + 9y^2 = 36$.

42. Prove that the circle having any focal distance of an ellipse as its diameter touches the auxiliary circle.

43. If P be any point on the ellipse whose foci are S, S', show that

$$\tan \tfrac{1}{2} PSS' \times \tan \tfrac{1}{2} PS'S = \frac{1-e}{1+e}.$$

[*If* $\angle PSS'$ *and* $PS'S$ *be* θ, *and* θ', *find* $\cos\theta$, $\cos\theta'$ *by applying the formula* $2bc \cos A = b^2 + c^2 - a^2$ *to the* $\triangle PSS'$.]

44. Find the condition that the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the two points where it is cut by the line $lx + my = 1$ should be perp. to each other.

[*The point of intersection of the tangents lies on the director circle.*]

45. An ellipse slides between two lines mutually at right angles. Show that the locus of its centre is a circle. [*C. U. 1932*]

9·20. Equation of a chord in terms of its middle point.

To find the equation of a chord of the ellipse $x^2/a^2 + y^2/b^2 = 1$ *in terms of its middle point* (x_1, y_1).

Any chord through (x_1, y_1) is

$$y - y_1 = m(x - x_1), \qquad \ldots \qquad \ldots \qquad (1)$$

or

$$y = mx + c, \qquad \ldots \qquad \ldots \qquad (2)$$

where $c = y_1 - mx_1$.

The abscissæ of the points of intersection of (2) and the ellipse are given by the roots of

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1,$$

or of

$$(a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0.$$

If x', x'' be the roots of this equation,

$$x_1 = \frac{x'+x''}{2} = -\frac{a^2mc}{a^2m^2+b^2} = -\frac{a^2m(y_1-mx_1)}{a^2m^2+b^2}, \quad (3)$$

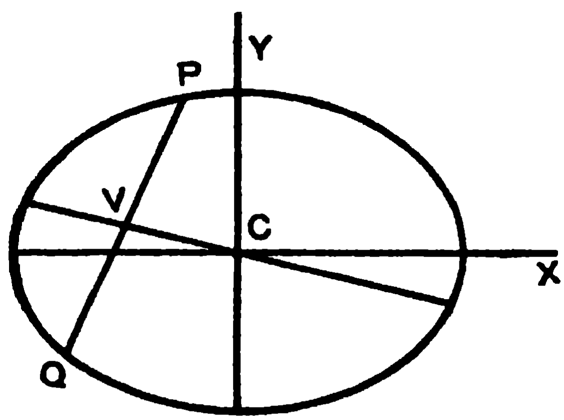
from which we get $\quad m = -b^2x_1/a^2y_1$.

Substituting this value in (1), we get the reqd. equation of the line as

$$(\mathbf{x}-\mathbf{x}_1)\frac{\mathbf{x}_1}{\mathbf{a}^2}+(\mathbf{y}-\mathbf{y}_1)\frac{\mathbf{y}_1}{\mathbf{b}^2}=\mathbf{0}.$$

9·21. Diameter.

To find the locus of the middle points of a system of parallel chords of the ellipse $x^2/a^2+y^2/b^2=1$.



Let the chords be parallel to

$$y = mx. \quad \cdots \quad (1)$$

Then any chord PQ of parallel system is

$$y = mx + c. \quad \cdots \quad (2)$$

c being different for different chords.

The abscissæ of the points of intersection P, Q of the ellipse and the line (2) are given by the roots of the equation

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

or $\quad (a^2m^2+b^2)x^2+2a^2mcx+a^2(c^2-b^2)=0. \quad (3)$

Let (x', y') be the mid-point V of PQ and x_1, x_2 the roots of (3).

$$\therefore \quad x' = \frac{x_1+x_2}{2} = -\frac{a^2mc}{a^2m^2+b^2},$$

$$y' = mx'+c = c-\frac{a^2m^2c}{a^2m^2+b^2} = \frac{b^2c}{a^2m^2+b^2}.$$

$\therefore \quad \dfrac{y'}{x'} = -\dfrac{b^2}{a^2 m} =$ constant (*i.e.* independent of c).

$\therefore$ V always lies on the line

$$y = -\frac{b^2}{a^2 m}x. \quad \cdots \quad \cdots \quad (4)$$

Hence the locus of the middle points of the chords of the ellipse drawn parallel to the line (1) is a line passing through the centre *viz.* $y = m'x$ where $m' = -b^2/a^2 m$ *i.e.* $mm' = -b^2/a^2$.

Def. The locus of the middle points of a system of parallel chords of an ellipse is called a *Diameter* and the chords are called its double ordinates.

Thus, *any diameter of the ellipse passes through its centre.*

9·22. Properties of Diameters.

(A) *The tangent at an extremity of any diameter is parallel to the chords which it bisects.*

The diameter bisecting chords parallel to

$$y = mx \quad \cdots \quad \cdots \quad (1)$$

is

$$y = -\frac{b^2}{a^2 m}x. \quad \cdots \quad \cdots \quad (2)$$

Let (x_1, y_1) be one of the points where (2) meets the ellipse,

then $\quad y_1 = -\dfrac{b^2}{a^2 m}x_1. \quad \therefore \ m = -\dfrac{b^2}{a^2}.\dfrac{x_1}{y_1}. \quad \cdots \quad (3)$

The tangent at (x_1, y_1) is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$,

or

$$y = -\frac{b^2}{a^2}\frac{x_1}{y_1}x + \frac{b^2}{y_1}$$

$$= mx + b^2/y_1 \quad \text{by (3)}$$

and hence it is parallel to (1).

(B) *The tangents at the ends of any chord meet on the diameter which bisects the chord.*

Let the chord PQ be

$$y = mx + c. \qquad \cdots \qquad \cdots \qquad (1)$$

Let (x_1, y_1) be the point of intersection of tangents at P, Q. Then PQ is the chord of contact of the tangents from (x_1, y_1) and hence its equation is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \qquad \cdots \qquad (2)$$

Since (1) and (2) represent the same line,

$$\therefore \quad m = -b^2x_1/a^2y_1.$$

Hence (x_1, y_1) lies on the line $y = -\dfrac{b^2}{a^2m}x$, which is the diameter bisecting the chord PQ.

9·23. Conjugate Diameters.

Two diameters are said to be **conjugate** *when each bisects all chords parallel to the other.*

From Art. 9·21, we see that the diameter bisecting chords parallel to the diameter $y = mx$ is $y = m'x$ where $mm' = -b^2/a^2$. The symmetry of the result shows that in this case the diameter $y = mx$ also bisects all chords parallel to the diameter $y = m'x$.

Thus, *the two diameters $y = mx$, $y = m'x$ will be conjugate if*

$$\mathbf{mm' = -\frac{b^2}{a^2}}.$$

9·24. Properties of conjugate Diameters.

(i) *The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.*

Let PCP', DCD' be a pair of conjugate diameters and let P be $(a \cos \varphi, b \sin \varphi)$ and D be $(a \cos \varphi', b \sin \varphi')$.

Equation to PCP' is $y = \frac{b}{a}\tan\varphi . x$.

Equation to DCD' is $y = \frac{b}{a}\tan\varphi' . x$.

Since the diameters are conjugate,

$$\therefore \quad \frac{b^2}{a^2}\tan\varphi \ \tan\varphi' = -\frac{b^2}{a^2},$$

$$\therefore \quad \tan\varphi\tan\varphi' = -1,$$

$$\text{or} \quad \cos\varphi\cos\varphi' + \sin\varphi\sin\varphi' = 0,$$

$$\text{or} \quad \cos(\varphi' - \varphi) = 0.$$

$$\therefore \quad \varphi' - \varphi = \pm\frac{\pi}{2}. \quad \ldots \quad \ldots \quad (1)$$

Cor. The values of φ' obtained from (1), viz. $\varphi + \frac{1}{2}\pi$, $\varphi - \frac{1}{2}\pi$ corresponding to φ, refer to the points D, D', respectively. Thus, *the co-ordinates of the four extremities of two conjugate diameters* are

$$P(a\cos\varphi,\ b\sin\varphi), \quad P'(-a\cos\varphi,\ -b\sin\varphi),$$

$$D(-a\sin\varphi,\ b\cos\varphi),\ D'(a\sin\varphi,\ -b\cos\varphi).$$

$\therefore$ *Equation to CP* is $y = \frac{b}{a}\tan\varphi . x$,

and *Equation to CD* is $y = -\frac{b}{a}\cot\varphi . x$.

(ii) *The sum of the squares of two conjugate semi-diameters of an ellipse is constant.*

Let PCP', DCD' be a pair of conjugate diameters.

Let P be $(a\cos\varphi, b\sin\varphi)$; then D is $\{a\cos(90^\circ + \varphi), b\sin(90^\circ + \varphi)\}$ *i.e.* $(-a\sin\varphi, b\cos\varphi)$.

$$\therefore \quad CP^2 = a^2\cos^2\varphi + b^2\sin^2\varphi,$$

$$CD^2 = a^2\sin^2\varphi + b^2\cos^2\varphi.$$

$$\therefore \quad \mathbf{CP^2 + CD^2 = a^2 + b^2} = \text{constant}. \quad \cdots \quad (2)$$

(iii) ***The tangents at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.***

Tangents at P, D, P', D' are respectively

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1, \qquad \frac{x}{a}\sin\varphi-\frac{y}{b}\cos\varphi=-1,$$

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=-1, \quad \frac{x}{a}\sin\varphi-\frac{y}{b}\cos\varphi=1.$$

$\therefore$ they form a parallelogram.

(iv) *The area of the parallelogram formed by the tangents at the ends of a pair of conjugate diameters of an ellipse is constant.*

Let EF, FG, GH, HE be the tangents at P, D, P', D'. Now, m's of CP, CD are $\dfrac{b\sin\varphi}{a\cos\varphi}$, $\dfrac{b\cos\varphi}{-a\sin\varphi}$. Hence they are par^l. to EH, EF ; *i.e.* $CPFD$ is a par^m.

$\therefore$ par^m. $EFGH=4$ par^m. $CPFD=8\triangle CPD$.

$$8\cdot\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a\cos\varphi & b\sin\varphi & 1 \\ -a\sin\varphi & b\cos\varphi & 1\end{vmatrix}=4ab.$$

Thus, *the area of the above parallelogram is equal to that of the rectangle formed by the tangents at the ends of the major and minor axes of the ellipse.*

(v) *If p be the perpendicular on the tangent at P from the centre* of an ellipse, then

$$\mathbf{p.CD=ab.}$$

Tangent at P is $\dfrac{x}{a}\cos\varphi+\dfrac{y}{b}\sin\varphi-1=0.$

$$p=\frac{1}{\sqrt{\dfrac{\cos^2\varphi}{a^2}+\dfrac{\sin^2\varphi}{b^2}}}=\frac{ab}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}=\frac{ab}{CD}.$$

Hence the result.

(vi) *The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point.*

Let the point P be $(a \cos \varphi, b \sin \varphi)$.

$$SP = a - ae \cos \varphi \qquad \text{[See Art. 9·5]}$$
$$S'P = a + ae \cos \varphi.$$

$$\therefore \quad SP.S'P = a^2 - a^2e^2 \cos^2\varphi$$
$$= a^2 - (a^2 - b^2) \cos^2\varphi$$
$$= a^2 \sin^2\varphi + b^2 \cos^2\varphi$$
$$= CD^2,$$

CD being the corresponding conjugate semi-diameter.

9·25. Equi-conjugate Diameters.

Def. When two conjugate diameters are equal, they are called *equi-conjugate diameters.*

Let P, D be the extremities of two equi-conjugate diameters. Then $CP^2 = CD^2$.

$$\therefore \quad a^2 \cos^2 \varphi + b^2 \sin^2 \varphi = a^2 \sin^2 \varphi + b^2 \cos^2\varphi.$$
$$\therefore \quad (a^2 - b^2) \cos^2 \varphi = (a^2 - b^2) \sin^2\varphi$$
$$\therefore \quad \tan^2\varphi = 1, \quad \text{or } \tan \varphi = \pm 1.$$
$$\therefore \quad \varphi = 45^\circ \text{ or } 135^\circ.$$

$$\therefore \quad \textit{Equation to CP is } y = \frac{b}{a} \tan \varphi . x = \pm \frac{b}{a} x. \quad \cdots \quad (1)$$

$$\textit{Equation to CD is } y = -\frac{b}{a} \cot \varphi . x = \mp \frac{b}{a} x. \cdots \quad (2)$$

Their joint equation is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$.

From (1) and (2), it is clear that the *two equi-conjugate diameters, CP, CD are equally inclined in opposite senses to the x-axis.*

Length of each equi-conjugate diameter

$$= 2CP = 2\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}$$
$$= 2\sqrt{\tfrac{1}{2}(a^2 + b^2)}, \quad \text{since } \cos^2 \varphi = \sin^2 \varphi = \tfrac{1}{2}.$$
$$= \sqrt{2(a^2 + b^2)}.$$

9·26. Supplemental chords.

Def. The chords joining any point on an ellipse to the extremities of a diameter are called *supplemental chords.*

To prove that supplemental chords are parallel to conjugate diameters.

Let P be any point $(a\cos\varphi,\ b\sin\varphi)$ on the ellipse, and let QCQ' be a diameter, the eccentric angles of whose end-points $Q,\ Q'$ are $\varphi_1,\ \varphi_1+\pi$.

The equations to $PQ,\ PQ'$ are [*by Art. 9·12*]

$$\frac{x}{a}\cos\frac{\varphi+\varphi_1}{2}+\frac{y}{b}\sin\frac{\varphi+\varphi_1}{2}=\cos\frac{\varphi-\varphi_1}{2}\quad\ldots(1)$$

$$\frac{x}{a}\cos\frac{\varphi+\varphi_1+\pi}{2}+\frac{y}{b}\sin\frac{\varphi+\varphi_1+\pi}{2}=\cos\frac{\varphi-\varphi_1-\pi}{2},$$

i.e. $$-\frac{x}{a}\sin\frac{\varphi+\varphi_1}{2}+\frac{y}{b}\cos\frac{\varphi+\varphi_1}{2}=\sin\frac{\varphi-\varphi_1}{2}.\quad\ldots(2)$$

The 'm' of line (1) $=-\dfrac{b}{a}\cot\dfrac{\varphi+\varphi_1}{2}$.

The 'm' of line (2) $=\dfrac{b}{a}\tan\dfrac{\varphi+\varphi_1}{2}$.

The product of m's $=-b^2/a^2$.

Hence the lines $PQ,\ PQ'$ are parallel to conjugate diameters.

9·27. Pair of tangents from a point.

To find the equation of a pair of tangents from $(x_1,\ y_1)$ to the ellipse.

Any line through $(x_1,\ y_1)$ is

$$y-y_1=m(x-x_1),\quad\ldots\quad(1)$$

or $$y=mx+y_1-mx_1.$$

If it touches the ellipse,

$$y_1 - mx_1 = \sqrt{a^2m^2+b^2}, \qquad [Art.\ 9{\cdot}11]$$

or $$(y_1 - mx_1)^2 = a^2m^2 + b^2. \qquad \ldots \qquad (2)$$

Now, eliminating m between (1) and (2), we get the equation of the pair of tangents from (x_1, y_1), *viz.*

$$\left(y_1 - \frac{y-y_1}{x-x_1}\cdot x_1\right)^2 = a^2\cdot\left(\frac{y-y_1}{x-x_1}\right)^2 + b^2,$$

or $$(xy_1 - yx_1)^2 = a^2(y-y_1{}^2) + b^2(x-x_1)^2.$$

This when simplified can be put in the form

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1{}^2}{a^2}+\frac{y_1{}^2}{b^2}-1\right) = \left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2.$$

9·28. Polar Equation.

(A) *To find the polar equation of an ellipse, referred to its centre as pole and major axis as the initial line.*

Putting $x = r\cos\theta$, $y = r\sin\theta$ in the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

we get $$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=1,$$

or $$\mathbf{\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}.$$

(B) *To find the polar equation of an ellipse referred to its focus as pole and major axis as the initial line.*

Let $P(r, \theta)$ be any point on the ellipse. Draw PM, PN perps. to the directrix and major axis.

Join SP and let l = semi-latus rectum SL. Now

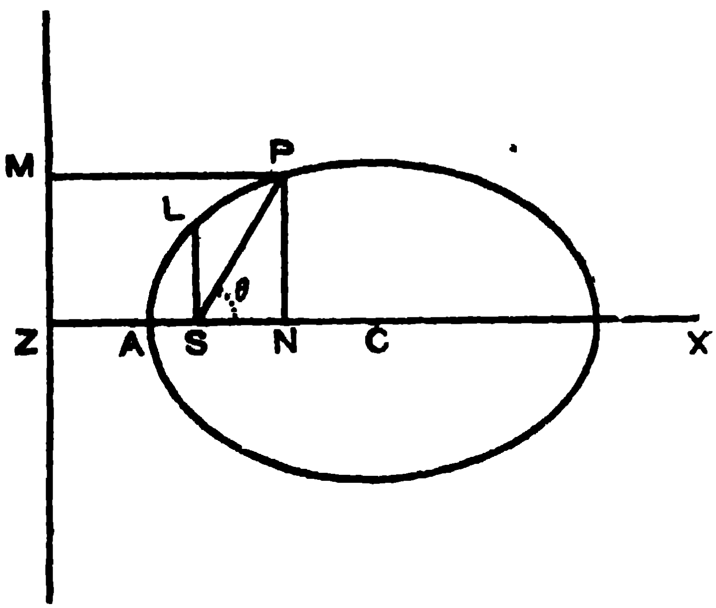
$$SL = eSZ,$$

$$r = SP = ePM = eZN = e(ZS+SN),$$

$$= e(SZ + SP\cos\theta) = l + er\cos\theta,$$

or $\quad r(1-e\cos\theta)=l.$

$$\therefore \quad r=\frac{l}{1-e\cos\theta} \qquad \ldots \qquad (2)$$



If SZ be taken as the positive direction of the initial line and $\angle ZSP=\theta$, then the equation becomes.

$$r=\frac{l}{1+e\cos\theta}$$

9·29. Important Geometrical properties.

We give below a collection of a few important geometrical properties which are of frequent occurrence. Some of these have been established before and some in Art. 9·30. [*Illustrative Examples*].

Here r, r' denote the focal distances of a point P on the ellipse, p, p' denote the perps. from the foci S, S' upon the tangent at P, p_0 is the perp. from the centre upon the tangent, CP, CD are two conjugate semi-diameters and α is the angle between them, and φ is the angle between either of the focal distances of P and the tangent at the point.

Then,

$$\mathbf{r+r'=2a} \quad \cdots \quad \cdots \quad (1)$$

$$\mathbf{CP^2+CD^2=a^2+b^2} \quad \cdots \quad \cdots \quad (2)$$

$$\mathbf{p.CD=ab} \quad \cdots \quad \cdots \quad (3)$$

$$\mathbf{CP.CD\ \sin\alpha=ab} \quad \cdots \quad \cdots \quad (4)$$

$$\mathbf{rr'=SP.SP'=CD^2} \quad \cdots \quad \cdots \quad (5)$$

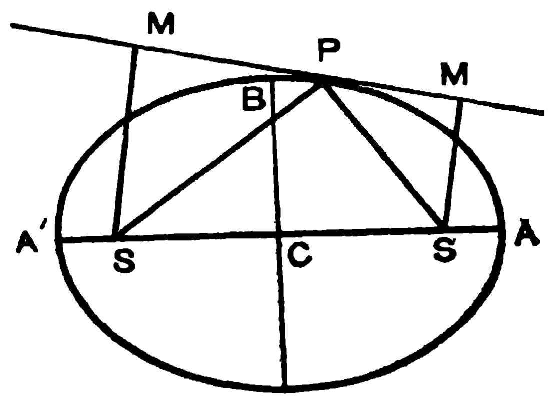
$$\frac{\mathbf{p}}{\mathbf{r}}=\frac{\mathbf{p'}}{\mathbf{r'}} \quad \cdots \quad \cdots \quad (6)$$

$$\mathbf{pp'=b^2} \quad \cdots \quad \cdots \quad (7)$$

$$\sin\phi=\frac{\mathbf{p}}{\mathbf{r}}=\frac{\mathbf{p'}}{\mathbf{r'}}=\frac{\mathbf{p_0}}{\mathbf{a}}. \quad \cdots \quad \cdots \quad (8)$$

9·30. Illustrative Examples.

Ex. 1. *The product of the perpendiculars from the two foci of an ellipse on the tangent at any point to it is equal to b^2, and the two focal distances of the point make equal angles with the tangent.*



Let p be the perp. from the focus S (ae, o) upon the tangent at $(a\cos\phi, b\sin\phi)$,

i.e. upon $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1=0$.

$$p=\frac{e\cos\phi-1}{\sqrt{\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}}}=\frac{ab(e\cos\phi-1)}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}. \quad (1)$$

Similarly, if p' be the perp. from the other focus $S'(-ae, o)$, upon the same tangent,

$$p'=\frac{ab(-e\cos\phi-1)}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}. \quad \cdots \quad (2)$$

$$\therefore \quad pp'=\frac{a^2b^2(1-e^2\cos^2\phi)}{a^2\sin^2\phi+b^2\cos^2\phi}.$$

Now, $a^2 \sin^2\phi+b^2 \cos^2\phi=a^2 \sin^2\phi+a^2(1-e^2) \cos^2\phi$

$$=a^2(1-e^2 \cos^2\phi).$$

$$\therefore \quad pp'=b^2.$$

Let $\qquad SP=r, \ S'P=r',$

then $\qquad r=SP=a-ex'=a(1-e \cos \phi) \qquad \ldots \qquad (3)$

$$r'=S'P=a+ex'=a(1+e \cos \phi). \qquad \ldots \qquad (4)$$

From (1), (2), (3), (4), it easily follows that

$$\frac{p}{r}=\frac{p'}{r'}.$$

$$\therefore \quad \angle SPM=\angle S'PM',$$

i.e. $\quad \Delta^s \ SPM, S'PM'$ are similar.

Note. Let p_0 be the perp. from the centre on the tangent at P and let ϕ (in the notation of Calculus) denote either of the angles SPM, $S'PM'$, then we have

$$\sin \phi=\frac{p}{r}=\frac{p'}{r'}=\frac{p+p'}{r+r'}=\frac{2p_0}{2a}=\frac{p_0}{a}.$$

This is an important relation.

Ex. 2. *If p be the perpendicular from the centre of an ellipse upon the tangent at any point P on it and r be the distance of P from the centre, show that*

$$a^2+b^2-r^2=\frac{a^2b^2}{p^2}.$$

If r' be the semi-diameter conjugate to the semi-diameter r, we have by Art. 9·24,

$$r^2+r'^2=a^2+b^2, \qquad \ldots \qquad (1)$$

and $\qquad pr'=ab. \qquad \ldots \qquad \ldots \qquad (2)$

Eliminating r' between (1) and (2), we get

$$r^2+a^2b^2/p^2=a^2+b^2,$$

or $\qquad a^2+b^2-r^2=a^2b^2/p^2.$

This is called the **pedal equation** of the ellipse *w. r. t. the centre.*

Ex. 3. *If p be the perpendicular from the focus upon the tangent to the ellipse at any point P and r be the distance of P from the focus, then*

$$\frac{\mathbf{b}^2}{\mathbf{p}^2}=\frac{2\mathbf{a}}{\mathbf{r}}-1.$$

Let r, r' be the focal distances of P and p, p' the perpendiculars from S, S' upon the tangent at P.

Then, we have $\qquad r+r'=2a \quad$ [*Art. 9·5*]

and $\qquad pp'=b^2. \quad$ [*Ex. 1.*]

Also, $\qquad \dfrac{p}{r}=\dfrac{p'}{r'}=\dfrac{b^2/p}{2a-r}=\dfrac{b^2}{p(2a-r)}.$

$$\therefore \quad \frac{b^2}{p^2}=\frac{2a-r}{r}=\frac{2a}{r}-1.$$

This is called the **pedal equation** of the ellipse *w. r. t. to the focus.*

Ex. 4. *Show that a pair of equi-conjugate diameters of an ellipse, may be designated either* (i) *as a pair of conjugate diameters, the sum of whose lengths is a* maximum or (ii) *as a pair of conjugate diameters whose mutual inclination is a* minimum.

(i) Let u, v be the lengths of semi-conjugate diameters.

$\therefore \quad u^2+v^2=k^2$ where $k^2=a^2+b^2$.

So $\quad (u+v)^2=2(u^2+v^2)-(u-v)^2=2k^2-(u-v)^2$

$\qquad\qquad =$ a maximum where $(u-v)^2$ is a minimum

i.e. when $\quad u=v$, *i.e.* when the diameters are equi-conjugate.

(ii) Let α be the inclination between two conjugate diameters $y=mx$ and $y=m'x$.

$$\therefore \quad \pm\tan\alpha=\frac{m-m'}{1+mm'}=\frac{m-m'}{1-b^2/a^2}=\frac{m-m'}{e^2}.$$

$$\therefore \quad m-m'=\pm e^2\tan\alpha.$$

$$\therefore \quad (m+m')^2=(m-m')^2+4mm'=e^4\tan^2\alpha-4b^2/a^2$$
$$=e^4\tan^2\alpha-4(1-e^2).$$

$$\therefore \quad e^4\tan^2\alpha-4(1-e^2)\geqslant 0.$$

$$\therefore \quad \tan^2\alpha\geqslant 4(1-e^2)/e^4,$$

$\therefore \quad |\tan\alpha|$ attains the minimum value $2\sqrt{1-e^2}/e^2$ when $m+m'=0$ *i.e.* $m=-m'$. But $mm'=-b^2/a^2$.

$$\therefore \qquad m=b/a, \qquad m'=-b/a.$$

$\therefore$ the equations of the two conjugate diameters become

$$y=\pm\frac{b}{a}x,$$

which are nothing but the equi-conjugate diameters.

Ex. 5. *Show that the angle between the tangents drawn from (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is*

$$=\tan^{-1}\frac{2ab\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1}}{x_1^2+y_1^2-a^2-b^2}.$$

We have seen in Art. 9·14, that the m's of the two tangents drawn from (x_1, y_1) are given by the roots of

$$(x_1^2-a^2)m^2-2x_1y_1m+y_1^2-b^2=0. \quad \cdots \quad (1)$$

Let α be the angle between the tangents and let m_1, m_2 be the two roots of (1).

Then, $$\tan\alpha=\frac{m_1-m_2}{1+m_1m_2}. \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

Now, $$(m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2$$
$$=\frac{4x_1^2y_1^2}{(x_1^2-a^2)^2}-4\frac{y_1^2-b^2}{x_1^2-a^2}$$
$$=\frac{4.\{x_1^2y_1^2-(x_1^2-a^2)(y_1^2-b^2)\}}{(x_1^2-a^2)^2}$$
$$=4\frac{b^2x_1^2+a^2y_1^2-a^2b^2}{(x_1^2-a^2)^2}$$
$$=\frac{4a^2b^2\left(\frac{x_1^2}{a_2}+\frac{y_1^2}{b_2}-1\right)}{(x_1^2-a^2)^2} \quad \cdots \quad \cdots \quad (3)$$

$$1+m_1m_2=1+\frac{y_1^2-b^2}{x_1^2-a^2}=\frac{x_1^2+y_1^2-a^2-b^2}{x_1^2-a^2} \quad \cdots \quad (4)$$

Now, extracting the square root of (3), we get the value of m_1-m_2 and dividing this by (4), the reqd. result is obtained from (2).

Examples IX(B)

1. Find the middle point of the chord of $x^2/a^2+y^2/b^2=1$ intercepted on the line $lx+my=1$.

2. Find the locus of the middle point of the chords of an ellipse

   (i) which pass through a fixed point (α, β), [*C. U.*]

   (ii) which subtend a right angle at the centre,

(iii) which are normal to the ellipse,

(iv) the tangents at the end of which intersect orthogonally.

3. Tangents are drawn from any point on the circle $x^2+y^2=a^2$ to $b^2x^2+a^2y^2=a^2b^2$; find the locus of the mid-point of the chord of contact.

4. Show that $4x-3y+4=0$ and $x+3y-7=0$ are parallel to the conjugate diameters of $4x^2+9y^2=36$.

5. Show that $3x^2+\lambda xy-4y^2=0$, (where λ is a parameter) represents a pair of conjugate diameters of $9x^2+16y^2=144$ for all values of λ.

6. If (x_1, y_1), (x_2, y_2) be the ends of conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$, then

(i) $x_1x_2/a^2+y_1y_2/b^2=0$;

and (ii) $x_2=\pm(a/b)y_1$, $y_2=\mp(b/a)x_1$.

7. If (x_1, y_1) and (x_2, y_2) be the extremities of a pair of conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$, show that $x_1y_2-x_2y_1=\pm ab$. [*C. U.*]

8. If the pair of lines $Ax^2+2Hxy+By^2=0$ be conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$, then $a^2A+b^2B=0$.

9. If the points of intersection of the ellipses $x^2/a^2+y^2/b^2=1$ and $x^2/\alpha^2+y^2/\beta^2=1$ be at the extremities of the conjugate diameters of the former, prove that

$$a^2/\alpha^2+b^2/\beta^2=2. \qquad [C.\ U.]$$

10. If the line $\frac{l}{a}x+\frac{m}{b}y=n$ cuts the ellipse $b^2x^2+a^2y^2=a^2b^2$ at the ends of conjugate diameters, then $l^2+m^2=2n^2$.

11. Show that the lines joining the centre of $x^2/a^2+y^2/b^2=1$ to its points of intersection with the line $y=x+\sqrt{\frac{1}{2}(a^2+b^2)}$, are a pair of conjugate diameters of the ellipse.

12. Show that the equation of the line joining the extremities of two conjugate diameters of the ellipse, the eccentric angle of one of which is φ, is

$$\frac{x}{a}(\cos\varphi - \sin\varphi) + \frac{y}{b}(\cos\varphi + \sin\varphi) = 1.$$

13. Prove that the chord, which joins the extremities of a pair of conjugate diameters of the ellipse, always touches a similar ellipse.

14. If P, Q are the extremities of conjugate diameters of an ellipse, find the locus of

(i) the middle point of PQ,

(ii) the foot of the perpendicular from the centre upon PQ,

(iii) the point of intersection of tangents at P, Q.

15. Show that the polar of any point on a diameter of the ellipse is parallel to its conjugate diameter.

16. Show that the major and minor axes are the *only* pair of conjugate diameters which are perpendicular to each other.

17. Prove that the lines joining the centre of an ellipse to the points of intersection of any tangent to the ellipse with its director circle, are conjugate diameters of the ellipse. [*C. U. 1936*]

18. For the ellipse $8x^2 + 12y^2 = 96$, find a pair of conjugate semi-diameters inclined at an angle $\tan^{-1}7$.

19. In an ellipse the tangents at the ends of its axes form a rectangle ; show that its diagonals are equi-conjugate diameters.

20. Prove that the product of two conjugate diameters is greatest when they are equi-conjugate.

21. Find the equation of the pair of tangents drawn from (4, 2) to the ellipse $x^2 + 2y^2 = 4$ and find the angle between them.

22. Find the equation of each of the tangents drawn from the point $(-15, -7)$ to the ellipse $9x^2+25x^2=225$.

[*C. U.*]

23. Find the locus of the point of intersection of a pair of tangents to the ellipse if

(i) $\tan\theta_1+\tan\theta_2=k$ (constant),

(ii) $\cot\theta_1+\cot\theta_2=\lambda$ (constant),

where θ_1, θ_1 are the angles which the tangents make with the major-axes.

24. A pair of common chords of the circle $x^2+y^2=a^2-b^2$ and the ellipse $x^2/a^2+y^2/b^2=1$ pass through the common centre. Find its equation.

[*Use* $S+\lambda S'=0$ *as representing a pair of common chords.*]

25. Find the common tangents of the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{a^2}+\frac{y^2}{\beta^2}=1.$$

[*Use the condition that* $lx+my+n=0$ *should touch both the ellipses ; hence obtain proportional values of* l, m, n.]

26. Verify that the three conics

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1, \quad \frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=1, \quad \frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=1$$

will have *a common tangent* if

$$\begin{vmatrix} a_1^2 & b_1^2 & 1 \\ a_2^2 & b_2^2 & 1 \\ a_3^2 & b_3^2 & 1 \end{vmatrix}=0.$$

27. Prove that the sum of the squares of the reciprocals of two perpendicular diameters of an ellipse is constant.

[*Use Art. 9·28, equation (1)*]

28. Show that the semi-latus-rectum of an ellipse is the harmonic mean between the segments of any focal chord.

[*Use Art. 9·28 equation* (2)]

29. Show that the sum of the reciprocals of any two perpendicular focal chords of an ellipse is constant.

30. If the square of the length of a diameter of the ellipse $4x^2 + 36y^2 = 144$ be

(1) the arithmetic mean, (2) the geometric mean,

or (3) the harmonic mean between the squares of the lengths of the major and minor axes, find the inclination of the diameter to the major axis in each case.

CHAPTER X

HYPERBOLA

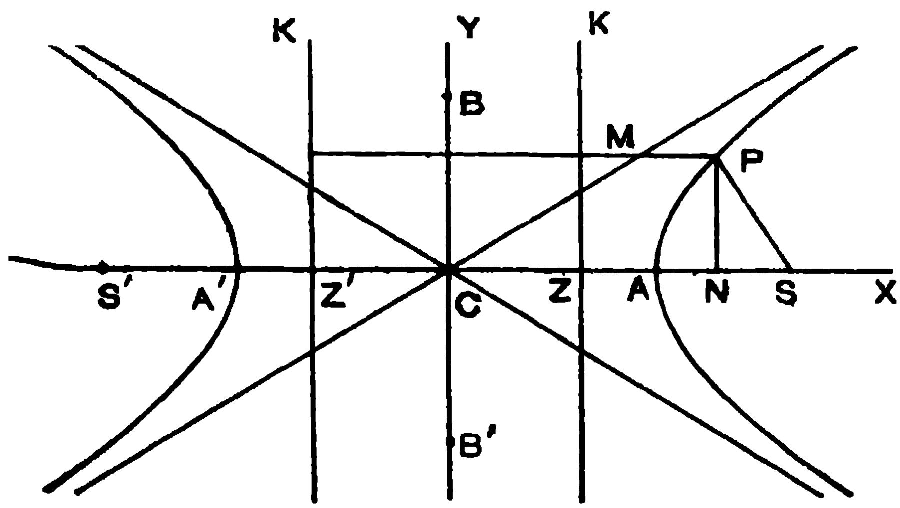
10·1. Definitions.

A *hyperbola* is the locus of a point which moves so that its distance from a fixed point bears a constant ratio, (greater than unity), to its distance from a fixed line.

The fixed point is called the *Focus*, the fixed line the *Directrix* and the ratio is called the *Eccentricity*, usually denoted by e ($>$ 1).

10·2. Standard Equation.

To find the equation to a hyperbola, given the focus, directrix and eccentricity.



Let S be the focus, ZK the directrix and e ($>$1) be the eccentricity. From S draw SZ perp. to ZK and divide SZ internally at A and externally at A' in the ratio $e : 1$, so that

$$\frac{SA}{AZ}=\frac{SA'}{A'Z}=e.$$

Let C be the mid-point of AA'. Draw CY perp. to AA'. Take C as origin and CAX, CY as x-axis and y-axis respectively Let $AA'=2a$; then $CA=CA'=a$, we have

$$e=\frac{SA}{AZ}=\frac{SA'}{A'Z}=\frac{CS-CA}{CA-CZ}=\frac{CS+CA'}{CA'+CZ}$$

$$=\frac{CS-a}{a-CZ}=\frac{CS+a}{a+CZ}$$

$$=\frac{2CS}{2a}=\frac{2a}{2CZ}.$$

$$CS=ae. \qquad \cdots \qquad \cdots \qquad (1)$$

$$CZ=a/e. \qquad \cdots \qquad \cdots \qquad (2)$$

$\therefore$ focus is (ae, o) and directrix is $x-a/e=0$.

Let (x, y) be the co-ordinates of any point P on the hyperbola. Draw PN perp. to AA' and PM perp. to the directrix. From the definition

$$SP=ePM,$$

or $\quad SP^2=e^2PM^2,$

or $\quad (x-ae)^2+y^2=e^2(x-a/e)^2,$

i.e. $\quad x^2(e^2-1)-y^2=a^2(e^2-1),$

i.e. $$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1. \qquad \cdots \qquad (3)$$

Since here $e>1$, $a^2(e^2-1)$ is positive. So we can set

$$\mathbf{b^2=a^2(e^2-1)} \qquad \cdots \qquad (4)$$

where b is a certain real positive constant.

$\therefore$ (3) becomes

$$\mathbf{\frac{x^2}{a^2}-\frac{y^2}{b^2}\ \ 1,} \qquad \cdots \qquad (5)$$

which is the *standard equation* of the hyperbola.

Note 1. If we take a point B' on the negative side of the origin such that $CS'=CS=ae$ and another point Z', such that $CZ'=CZ=a/e$

and draw $Z'K'$ perp. to AA', PM' perp. to $Z'K'$, then it can be easily shown from the equation (5), that the relation $S'P^2 = e^2 PM'^2$ is satisfied and hence S' is the *second focus*, and $Z'K'$, is the *second directrix*. Thus *the hyperbola has a second focus and a second directrix*.

Note 2. The points A, A' are called the *Vertices* of the hyperbola, C, its *Centre*; the line ACA' is called the *Transverse Axis*, and the line BB' is called the *Conjugate Axis*, where B and B' are the points on the y-axis such that $CB = CB' = b$. It should be noted, as is clear from the figure, that the conjugate axis $x=0$ meets the hyperbola $x^2/a^2 - y^2/b^2 = 1$ in imaginary points namely $y = \pm ib$.

It should also be noted that in the hyperbola a may be $>$ or $<$ b.

Note 3. *Peculiarities of the curve.*

(i) As we have seen in the case of the ellipse, here also the curve is symmetrical about the origin, (which is therefore called centre), and also symmetrical about either axis.

(ii) Since $y^2 = \dfrac{b^2}{a^2}(x^2 - a^2)$, it follows that for any value of x lying between $+a$ and $-a$, y is imaginary, but for other values of x outside these limits, y is real. This shows that *no part of the curve can lie between the lines $x = \pm a$.*

(iii) Since $x^2 = \dfrac{a^2}{b^2}(y^2 + b^2)$, it follows that for all real values of y however great, positive or negative, x has a real value. This shows that *the curve will consist of two symmetrical branches, each extending to infinity in two directions*, as shown in the figure.

(iv) Transforming equation (5) to polar co-ordinates,

we have

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = \frac{\cos^2\theta}{b^2}\left(\frac{b^2}{a^2} - \tan^2\theta\right). \quad \cdots \qquad (1)$$

So long as $\tan^2\theta < b^2/a^2$, equation (1) gives two equal opposite values of r corresponding to any value of θ. But when $\tan^2\theta > b^2/a^2$ *i.e.* when $\tan\theta$ does not lie between $-b/a$ and b/a, r^2 is negative and hence r is imaginary. This shows that *the hyperbola lies entirely between the two lines drawn from the centre, making angles $\tan^{-1} b/a$ with the transverse axis.*

Note 4. With reference to the above equation of a hyperbola the following results should be noted :

(i) Co-ordinates of the **centre (0, 0)**

(ii) Co-ordinates of the **focus** : **($\pm$ae, o).**

(iii) Equation of **Transverse axis** : $y=0$.

(iv) Equation of **Conjugate axis** ; $x=0$.

(v) Equation of **Directrices** : $x=\pm a/e$.

(vi) **Eccentricity** $\mathbf{e^2=\frac{a^2+b^2}{a^2}=1+\left(\frac{conj.\ axis}{transv.\ axis}\right)^2}$.

(vii) **Latus rectum** $=\mathbf{\frac{2b^2}{a}}$.

(viii) **Length of transverse axis = 2a.**

(ix) **Length of conjugate axis = 2b.**

10·3. General Equation.

To find the equation of a hyperbola whose focus is (α, β), *whose directrix* $lx+my+n=0$ *and eccentricity is* e (>1).

Let $P(x, y)$ be any point on the hyperbola, S be the focus, and PM be the perp. on the directrix.

$$\therefore \qquad SP=ePM,$$

$$\text{or} \qquad SP^2=e^2PM^2.$$

$$\therefore\ (x-\alpha)^2+(y-\beta)^2=e^2\cdot\frac{(lx+my+n)^2}{l^2+m^2}.$$

This is the reqd. eq^n.

10·4. Latus rectum.

The *Latus rectum* is the chord through either focus perp. to the transverse axis.

Let LSL' be a latus rectum and let the co-ordinates of L be (ae, l) so that $SL=l$.

$$\therefore\ \frac{a^2e^2}{a^2}-\frac{l^2}{b^2}=1,\ \text{ or }\ \frac{l^2}{b^2}=e^2-1=\frac{b^2}{a^2}.$$

$$\therefore\ l^2=\frac{b^4}{a^2}.\quad \therefore\ l=\frac{b^2}{a}.$$

$$\textit{Latus rectum}=\mathbf{\frac{2b^2}{a}}.$$

10·5. Focal Distances of a point.

Let (x_1, y_1) be the co-ordinates of P.

Then $\quad SP = ePM = eNZ = e(CN - CZ) = ex_1 - a,$

$\quad S'P = ePM' = eNZ' = e(CN + CZ') = ex_1 + a.$

$\therefore \quad S'P - SP = (ez_1 + a) - (ex_1 - a) = 2a.$

Hence *the difference of the focal distances of any point on the hyperbola is constant and is equal to the transverse axis.*

10·6. Parametric representation.

Plainly
$$\mathbf{x} = \mathbf{a} \sec \varphi$$
$$\mathbf{y} = \mathbf{b} \tan \varphi$$

satisfy the equation of the hyperbola whatever be the value of φ. These may therefore be taken as the co-ordinates of *any* point on the hyperbola, φ being the parameter.

Also
$$\mathbf{x} = \mathbf{a} \cosh \phi$$
$$\mathbf{y} = \mathbf{b} \sinh \phi$$

may be taken as the co-ordinates of *any* point on the hyperbola, since they satisfy the equation of the hyperbola.

10·7. Corresponding Formulæ.

Many of the results obtained in connection with the ellipse in the previous chapter hold true for the hyperbola merely by putting $-b^2$ for b^2 in the corresponding results of the ellipse. They can be established exactly in the same way as in the case of the ellipse.

For example,

the condition that $y = mx + c$ shall touch the hyperbola is
$$c = \pm \sqrt{a^2m^2 - b^2},$$

the equation of the tangent at (x_1, y_1) is
$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1,$$

the equation of the director circle is $x^2 + y^2 = a^2 - b^2$.

As in the case of the ellipse, the auxiliary circle is the circle described upon the transverse axis as diameter and hence its equation is $x^2+y^2=a^2$.

10·8. An Important Lemma.

If α, β be the roots of the quadratic in x,

$$\text{viz.} \qquad ax^2+bx+c=0, \qquad \cdots \qquad \cdots \qquad (1)$$

the quadratic in y (obtained by writing $1/y$ for x)

$$\text{viz.} \quad a\frac{1}{y^2}+b\frac{1}{y}+c=0, \quad \text{i.e.} \quad cy^2+by+a=0 \qquad \cdots \qquad (2)$$

will have for its roots $1/\alpha$, $1/\beta$.

Manifestly (2) will have *one* zero root if $a=0$ and *two* zero roots if $a=0$ and $b=0$, So (1) will have one infinite root if $a=0$ and two infinite roots if $a=b=0$. In the latter case (when $a=b=0$) the condition for equal roots is automatically fulfilled.

In other words when the coefficients of the second and linear powers in a quadratic equation vanish, so that the equation assumes the symbolic form

$$0.x^2+0.x+\text{const.}=0,$$

the two roots are equal, their common value being ∞.

10·9. Asymptotes.

An *Asymptote* of a curve is a straight line which meets it in two points at infinity but which is not altogether at infinity.

To find the asymptotes of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The line $y=mx+c$ $\cdots$ (1) meets the hyperbola in points whose abscissæ are given by

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1,$$

i.e. by $(b^2-a^2m^2)x^2-2a^2mcx-a^2(c^2+b^2)=0. \quad \cdots \quad (2)$

If (1) be an asymptote of the curve, both the roots of (2) will be infinite, which requires that

$$b^2 - a^2m^2 = 0, \qquad 2a^2mc = 0.$$

Hence $\qquad m = \pm b/a, \qquad c = 0.$

$\therefore$ the equations of the asymptotes are

$$\mathbf{y} = \pm \frac{\mathbf{b}}{\mathbf{a}}\mathbf{x} \qquad \cdots \qquad \cdots \qquad (3)$$

or (written as one equation)

$$\frac{\mathbf{x}^2}{\mathbf{a}^2} - \frac{\mathbf{y}^2}{\mathbf{b}^2} = 0. \qquad \cdots \qquad \cdots \qquad (4)$$

Cor. 1. From the above equations of *the asymptotes*, it is clear that they *pass through the centre* of the curve and *the axes of the conic are the angle-bisectors of the asymptotes.*

Cor. 2. If in the above result, $m = \pm b/a$ but $c \neq 0$, then the lines $y = \pm \frac{b}{a}x + c$ will cut the hyperbola in one point at infinity and they *are parallel to the asymptotes.*

10·10. Asymptotes by Inspection.

If the equation of a hyperbola can be put in the form

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) - k = 0 \qquad \cdots \qquad (1)$$

where k is a constant, then each of the lines

$$a_1x + b_1y + c_1 = 0$$

and $a_2x + b_2y + c_2 = 0$ *is an asymptote.*

For take the line $\quad a_1x + b_1y + c_1 = 0,$

$$\text{or} \quad y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1}. \qquad \cdots \qquad \cdots \qquad (2)$$

If we eliminate y between (1) and (2), to obtain the abscissæ of their common points of intersection, it is

obvious from the form of the equation (1) that the quadratic in x, giving the abscissæ, would at once reduce to the form

$$0.x^2+0.x-k=0 \qquad \cdots \qquad \cdots \quad (3)$$

since the first (product) term of (1) would be zero.

Hence (x_1, y_1), (x_2, y_2) being the points of intersection of (1) and (2).

$$x_1=x_2=\infty$$

and since $\qquad y_1=mx_1+c,\ y_2=mx_2+c,$

$$\therefore \quad y_1=y_2=\infty.$$

$\therefore \quad a_1x+b_1y+c_1=0$ is an asymptote.

Similarly, $a_2x+b_2y+c_2=0$ is also an asymptote.

Cor. 1. Thus, whenever the equation of a conic is of the form

$$LM=\text{constant},$$

where L, M are linear functions, the two asymptotes of the conic are

$$L=0 \text{ and } M=0.$$

Conversely, the equation of any hyperbola having for its asymptotes $\qquad L=0$, $M=0$, can be written as

$$LM=k,$$

where k is a variable parameter.

From above it is clear that *the joint equation of the asymptotes differs from that of the hyperbola by a constant.*

10·11. Conjugate Hyperbola.

A hyperbola S' whose transverse axis is BB' $(=2b)$ and conjugate axis is AA' $(=2a)$ is a called a *Conjugate Hyperbola.* It is conjugate to

$$S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0. \qquad \cdots \quad (1)$$

Its equation referred to the same set of axes is

$$S' \equiv \frac{y^2}{b^2} - \frac{x^2}{a^2} - 1 = 0.$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \qquad (2)$$

We easily see that the equation of S' differs from that of S in having $-a^2$ for a^2 and $-b^2$ for b^2.

(1) If *e be the eccentricity of the conjugate hyperbola*

$$e^2 = 1 + \frac{a^2}{b^2} = \frac{a^2 + b^2}{a^2}.$$

(2) *A hyperbola and its conjugate have the same centre and the same asymptotes.*

(3) *Foci of the conic* are $(0, \pm ae)$.

(4) *Directrices of the conic* are $y = \pm a/e$.

Note 1. *Peculiarities of the curve.*

From the equation of S', it can be shown as in the case of S (Art. 10·2, Cor. 3) that no part of S' can lie between the lines $y = \pm b$, the curve consists of two symmetrical branches, each extending to infinity in two directions and it lies entirely in two sections formed by the asymptotes in which S does not lie.

Note 2. *A hyperbola and its conjugate cannot intersect in real points, as their equations are algebraically inconsistent.*

10·12. Conjugate Diameters.

(i) *If a pair of diameters be conjugate with respect to a hyperbola, they will also be conjugate with respect to its conjugate hyperbola.*

The lines $y = mx$ and $y = m'x$ are conjugate *w. r. t.* the hyperbola S if $mm' = b^2/a^2$ (Art. 10·11). Now the condition that the above two diameters will be conjugate *w. r. t.* S' is $mm' = -b^2/-a^2 = b^2/a^2$, which is the same as before. Hence the result

(ii) *If a pair of diameters be conjugate w. r. t. a hyperbola, one of them meets the hyperbola in two real points and the other meets the hyperbola in two imaginary points and the conjugate hyperbola in real points.*

Let $y=mx$ and $y=m'x$ be the diameters ; then

$$mm'=b^2/a^2.$$

Suppose $\quad |m| < b/a, \quad$ then $|m'| > b/a.$

These two diameters meet the hyperbola S in points whose abscissæ are given respectively by

$$x^2\left(\frac{1}{a^2}-\frac{m^2}{b^2}\right)=1 \text{ and } x^2\left(\frac{1}{a^2}-\frac{m'^2}{b^2}\right)=1.$$

The first equation gives *real* values of x and the second equation gives *imaginary* values of x.

Similarly, the abscissæ of the points of intersection of the above two lines with the conjugate hyperbola S' are given by

$$x^2\left(\frac{1}{a^2}-\frac{m^2}{b^2}\right)=-1 \text{ and } x^2\left(\frac{1}{b^2}-\frac{m'^2}{b^2}\right)=-1.$$

In the first case the roots are *imaginary* and in the second case the roots are *real.*

Hence the theorem.

Note 1. It should be noted that when we speak of (PP', DD') as a pair of conjugate diameters of the hyperbola S, we mean in reality that one of them is a chord of S and that the other (*i. e.* the conjugate line) is a chord of the conjugate hyperbola S'.

(iii) *If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D, then*

$$CP^2-CD^2=a^2-b^2.$$

Let P $(a \sec \varphi, b \tan \varphi)$ be any point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \quad \dots \qquad \dots \quad (1)$$

Equation of the diameter CP is

$$y=\frac{b \tan \phi}{a \sec \varphi} \cdot x=\frac{b}{a} \sin \varphi . x . \quad \ldots \quad (2)$$

Equation of the diameter CD, conjugate to CP is

$$y=\frac{b}{a \sin \varphi} \cdot x \quad \ldots \quad (3) \qquad [\because mm'=b^2/a^2]$$

Line (3) meets the conjugate hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1, \quad \ldots \quad \ldots \quad (4)$$

in points given by

$$\frac{x^2}{a^2}\left(1-\frac{1}{\sin^2 \varphi}\right)=-1, \text{ i.e. } \frac{x^2}{a^2}\left(1-\operatorname{cosec}^2 \varphi\right)=-1.$$

Hence, $x= \pm a \tan \varphi, \ y= \pm b \sec \varphi$.

$\therefore$ D is the point $(\pm a \tan \varphi, \pm b \sec \varphi)$.

$\therefore$ $CP^2=a^2 \sec^2 \varphi+b^2 \tan^2 \varphi$,

$CD^2=a^2 \tan^2 \varphi+b^2 \sec^2 \varphi$.

$\therefore$ $CP^2-CD^2=(a^2-b^2)(\sec^2 \varphi-\tan^2 \varphi)=a^2-b^2$.

Cor. Thus, we see that if the points P, P' of a diameter of the hyperbola are $(a \sec \varphi, b \tan \varphi)$, $(-a \sec \varphi, -b \tan \varphi)$ then the end points D, D' (lying on the conjugate hyperbola) of the conjugate diameter will be $(a \tan \varphi, b \sec \varphi)$, $(-a \tan \varphi, -b \sec \varphi)$. All the four points are *real.*

Note 2. Let D_1, D_2 be the points (imaginary) where the conjugate diameter CD meets the hoperbola. Then D_1, D_2 are $(ia \tan \phi, ib \sec \phi)$, $(-ia \tan \phi, -ib \sec \phi)$.

Now, $CP^2=a^2 \sec^2 \phi+b^2 \tan^2 \phi$.

$CD_1{}^2=-a^2 \tan^2 \phi-b^2 \sec^2 \phi$.

$\therefore$ $CP^2+CD_1{}^2=(a^2-b^2)(\sec^2 \phi-\tan^2 \phi)$

$=a^2-b^2$.

(iv) *If CP, CD be two conjugate semi-diameters of the hyperbola of which S and S' are the foci then* $SP.\ S'P=CD^2$.

Let P be the point $(a \sec \varphi, b \tan \varphi)$.

$\therefore$ by Art. 10·5, $SP = ex_1 - a = e.a \sec \varphi - a$,

$$S'P = ex_1 + a = ea \sec \varphi + a.$$

$$\therefore \quad SP\, S'P = a^2(e \sec \varphi - 1)(e \sec \varphi + 1)$$
$$= a^2(e^2 \sec^2 \varphi - 1)$$
$$= (a^2 + b^2) \sec^2 \varphi - a^2$$
$$= a^2 \tan^2 \varphi + b^2 \sec^2 \varphi$$
$$= CD^2.$$

(v) *If a pair of conjugate diameters meet the hyperbola in P, P' and its conjugate in D, D', then the tangents at P, P', D., D' form a parallelogram whose vertices lie on the asymptotes and whose area is constant and equal to $4ab$.*

The tangents at $P(a \sec \varphi, b \tan \varphi)$ and $D(a \tan \varphi, b \sec \varphi)$ to the hyperbola and its conjugate are respectively

$$\frac{x}{a} \sec \varphi - \frac{y}{b} \tan \varphi = 1 \qquad \ldots \quad (1)$$

$$\frac{x}{a} \tan \varphi - \frac{y}{b} \sec \varphi = -1. \qquad \ldots \quad (2)$$

Lines (1) and (2) are obviously parallel to CD and CP. For the point of intersection of (1) and (2) we add their equations, and get

$$\frac{x}{a} = \frac{y}{b}, \; i.e. \; y = \frac{b}{a} x.$$

i.e. the point of intersection of the tangents at P and D, (say E) lies on the asymptote

$$y = (b/a)x.$$

Similarly, tangents at D and P' meet at F on the asymptote $\qquad y = (-b/a)x,$

tangents at P' and D' meet at G on the asymptote

$$y = (b/a)x,$$

tangents at D' and P meet at H on the asymptote

$$y = (-b/a)x.$$

Now the four tangents at P, P', D, D' form the par^m. $EPGH$,

which is obviously $= 4$ par^m. $CPED$

$$= 8\,\triangle CPD$$

$$= 8 \times \tfrac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a\sec\varphi & b\tan\varphi & 1 \\ a\tan\varphi & b\sec\varphi & 1 \end{vmatrix}$$

$$= 4ab.$$

10·13. Relation connecting equations of a hyperbola, its asymptotes and its conjugate.

The equations of the hyperbola S, its asymptotes A, and its conjugate S' are

$$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

$$A \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

$$S' \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0.$$

Since the terms $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ common to these equations will, for any change of origin and axes, change into terms which will be the same for all three, the above three equations will always be connected by the relation

$$S' - A = A - S,$$

i.e. $\mathbf{S + S' = 2A.}$

10·14. Rectangular (or Equilateral) Hyperbola.

Def. A *Rectangular Hyperbola* is defined as a hyperbola whose asymptotes are perpendicular to each other.

The asymptotes of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ being

$y = \pm \frac{b}{a} x$, they will be perp. to each other if $\frac{b}{a} \times \left(-\frac{b}{a}\right)$ $= -1$, *i.e.* if $\frac{b^2}{a^2} = 1$, *i.e.* if $a^2 = b^2$, *i.e.* if $a = b$.

The *equation of the rectangular hyperbola referred to its axes as axes of co-ordinates* is therefore

$$\mathbf{x^2 - y^2 = a^2} \qquad \dots \quad (1)$$

and that of its *conjugate* is

$$x^2 - y^2 = -a^2. \qquad \dots \quad (2)$$

The *asymptotes* are

$$\mathbf{y = \pm x.} \qquad \dots \qquad \dots \quad (3)$$

Thus, *the asymptotes are the angle-bisectors of the axes ; and conversely, the axes also are the angle-bisectors of the asymptotes.*

Since, $e^2 = \frac{a^2 + b^2}{a^2} = 2$ when $a = b$,

the *eccentricity* of every rectangular hyperbola is

$$\mathbf{e = \sqrt{2}.}$$

Since the equation of the Director circle is

$$x^2 + y^2 = a^2 - b^2 = 0 \text{ when } a = b,$$

the *Director circle* of a rectangular hyperbola is a *point circle.*

The diameters $y = mx$, $y = m'x$ will be conjugate w. r. t. the rectangular hyperbola if

$$\mathbf{mm'} = b^2/b^2 = 1, \quad i.e. \text{ if } m' = 1/m.$$

Hence the lines $y = mx$, $y = \frac{1}{m} x$ are *a pair of conjugate diameters.*

Def. An *Equilateral Hyperbola* is defined as a hyperbola in which the transverse axis is equal to the conjugate axis (*i.e.* $a = b$). Hence every equilateral hyperbola is a rectangular hyperbola and conversely.

10·15. Rectangular Hyperbola referred to its asymptotes as axes of co-ordinates.

The equation of the hyperbola referred to its axes as axes of co-ordinates is

$$x^2 - y^2 = a^2. \quad \ldots \qquad \ldots \quad (1)$$

Keeping the origin fixed, turn the axes of co-ordinates through an angle of $-45°$; then we have to substitute

$$x \cos 45° + y \sin 45°, \ \textit{i.e.} \ \frac{x+y}{\sqrt{2}} \text{ for } x$$

$$-x \sin 45° + y \cos 45°, \ \textit{i.e.} \ \frac{y-x}{\sqrt{2}} \text{ for } y$$

in the equation (1).

Thus, equation (1) transforms into

$$\left(\frac{x+y}{\sqrt{2}}\right)^2 - \left(\frac{y-x}{\sqrt{2}}\right)^2 = a^2,$$

or $\qquad xy = \frac{1}{2}a^2.$

which can be written as

$$\mathbf{xy = c^2}$$

where $\quad c^2 = \frac{1}{2}a^2.$

This is the required equation.

Note. The *conjugate hyperbola* will obviously have the equation $xy = c^2$, if the same substitutions are made for x, y in the equation $-a^2$.

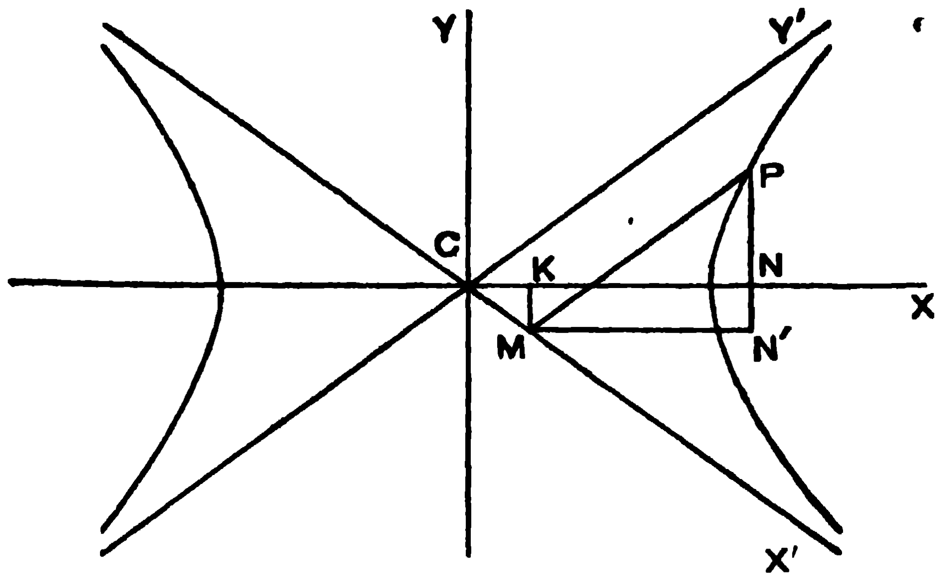
10·16. Hyperbola referred to its asymptotes as axes of co-ordinates.

Let (x, y) be the co-ordinates of any point P on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots \qquad \ldots \quad (1)$$

w. r. t. its axes as axes of co-ordinates.

Let (x', y') be the co-ordinates of P *w. r. t.* its asymptotes CX', CY' as axes of reference, 2α being the angle between them.



Draw PM parallel to CY' meeting CX' in M. Then

$$x' = CM, \; y' = PM.$$

Since the asymptote CY' is inclined at an angle of $\tan^{-1} b/a$ with the transverse axes,

$$\alpha = \tan^{-1} b/a, \qquad \therefore \quad \tan \alpha = b/a.$$

$$\sin \alpha = \frac{b}{\sqrt{a^2+b^2}}, \; \cos \alpha = \frac{a}{\sqrt{a^2+b^2}}.$$

Draw PN, MK perp. to CX and through M draw MN' parallel to CX to meet PN produced at N'.

Then, $\qquad x = CN, \; y = PN.$

$$x = CN = CK + KN = CK + MN' = CM \cos \alpha + MP \cos \alpha,$$

$$= (x' + y') \cos \alpha = (x' + y') \frac{a}{\sqrt{a^2+b^2}};$$

$$y = PN = PN' - NN' = PN' - MK = MP \sin \alpha - CM \sin \alpha$$

$$= (y' - x') \sin \alpha = (y' - x') \frac{b}{\sqrt{a^2+b^2}}.$$

Since (x', y') satisfy the equation (1),

$$\therefore \quad (x' + y')^2 - (y' - x')^2 = a^2 + b^2,$$

$$\text{or} \qquad x'y' = \tfrac{1}{4}(a^2 + b^2).$$

Hence the required equation is

$$xy = \tfrac{1}{2}(a^2 + b^2) \qquad (1)$$

or
$$\mathbf{xy = c^2} \qquad \cdots \qquad (2)$$

where $c^2 = \tfrac{1}{2}(a^2 + b^2)$.

Note 1. The equation of the *conjugate hyperbola* when referred to the asymptotes as axes of reference can be found similarly as before to be
$$xy = -c^2. \qquad \cdots \qquad \cdots \qquad (3)$$

Note 2. It should be noted that although this form of the equation is the same as that obtained in the previous article, ***here the axes of reference are oblique***, but there they are rectangular.

10·17. On the Hyperbola $xy = c^2$.

(i) *The chord joining two points* (x_1, y_1), (x_2, y_2).

The equation of the chord is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \qquad \cdots \qquad (1)$$

Since the points lie on $xy = c^2$,

$$\therefore \quad x_1 y_1 = c^2,\ x_2 y_2 = c^2. \qquad \cdots \qquad (2)$$

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{c^2/x_1 - c^2/x_2}{x_1 - x_2} = -\frac{c^2}{x_1 x_2}.$$

$\therefore$ (1) becomes

$$y - y_1 = -\frac{c^2}{x_1 x_2}(x - x_1). \qquad \cdots \qquad (3)$$

(ii) *Tangent at* (x_1, y_1).

Putting $x_2 = x_1$ in (3), we find the equation of the tangent at (x_1, y_1) to be

$$y - y_1 = -\frac{c^2}{x_1^2}(x - x_1) = -\frac{y_1}{x_1}(x - x_1) \quad \text{by (2)}$$

or
$$yx_1 + xy_1 = 2x_1 y_1,$$

or
$$\mathbf{\frac{x}{x_1} + \frac{y}{y_1} = 2.}$$

The above results are true whether the hyperbola is rectangular or not.

10·18. Parametric representation of $xy = c^2$.

Since
$$x = ct$$
$$y = \frac{c}{t}$$

satisfy the equation $xy = c^2$, whatever be the value of t, they can be taken to represent the co-ordinates of any point on the hyperbola $xy = c^2$, rectangular or otherwise t being the variable parameter.

As in the previous article,

(i) *the chord joining the points* 't_1' 't_2' is
$$x + yt_1t_2 = c(t_1 + t_2)$$

(ii) *the tangent at* 't' is
$$x/t + yt = 2c.$$

10·15. Illustrative Examples.

Ex. 1. *Find the asymptotes of the curve*
$$2x^2 + 5xy + 2y^2 + 4x + 5y = 0. \quad [C.\ U.\ 1935,\ '40]$$

Since the joint equation of the asymptotes differs from the equation of the hyperbola by a constant, the equation of the asymptotes will be
$$2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0.$$

Since it would represent a pair of lines, hence by using the well-known criterion for a pair of lines, we have
$$2.2.\lambda + 2.\tfrac{5}{2}.2.\tfrac{5}{2} - 2.\tfrac{25}{4} - 2.4 - \lambda.\tfrac{25}{4} = 0,$$
or $\quad \frac{9}{4}\lambda = \frac{9}{2}. \qquad \therefore \ \lambda = 2.$

$\therefore$ the equation of the asymptotes is
$$2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0, \qquad \dots \qquad (1)$$
or $$2x^2 + x(5y + 4) + 2y^2 + 5y + 2 = 0.$$

Solving it as a quadratic in x, we have
$$x = \frac{-5y - 4 \pm 3y}{4},$$

or $$x+2y+1=0,\ 2x+y+2=0,$$
which are the separate equations of the aspmptotes.

Ex. 2. *The area of a triangle cut off from the asymptotes by a tangent to the hyperbola is constant.* [*C. U. 1930, '34, '38, '44*]

Let the hyperbola be $xy=c^2$, the asymptotes being the axes of reference and let $2a$ be the angle between the asymptotes. Then $\tan a=b/a$.

Tangent at any point $P(x_1, y_1)$ is

$$\frac{x}{x_1}+\frac{y}{y_1}=2. \qquad [Art.\ 10{\cdot}17]$$

Suppose the tangent intersects the asymptote CX in D and CY in E. Then D is $(2x_1, o)$ E is $(o, 2y_1)$ *i.e.* $CD=2x_1$, $CE=2y_1$.

$$\therefore\quad \Delta CDE=\tfrac{1}{2}\ CD.CE \sin DCE=\tfrac{1}{2}.\ 2x_1 2y_1.\ \sin 2a$$
$$=2c^2 \sin 2a\ [\because\ (x_1, y_1) \text{ lie on the conic } xy=c^2],$$
$$=\text{constant}.$$

The actual value of the area can be obtained by substituting the values of $\quad c^2=\frac{1}{2}(a^2+b^2)$,

and $\quad \sin 2a=\dfrac{2 \tan a}{1+\tan^2 a}=\dfrac{2b/a}{1+b^2/a^2}=\dfrac{2ab}{a^2+b^2}.$

$$\therefore\quad \Delta CDE=2.\tfrac{1}{2}(a^2+b^2)\cdot\frac{2ab}{a^2+b^2}=ab.$$

Ex. 3. *The portion of the tangent at any point of a hyperbola intercepted between the axes is bisicted at the point of contact.*
[*C. U. 1932*]

Proceeding as in Ex. 2, we have $D(2x_1, o)$ and $E\ (o, 2y_1)$. Hence the mid-point of DE is (x_1, y_1)—which is P.

Hence the result.

Examples X

1. Find the equation to the hyperbola,

(i) whose focus is (2, 2), eccentricity 2 and directrix
$$x+y=9,$$

(ii) whose focus is (1, −8), eccentricity $\sqrt{5}$ and directrix $\qquad 3x-4y=10.$

2. Find the locus of a point which moves so that the difference of its distances from the points $(\pm 5, 0)$ is 6.

3. Show that the locus of the centre of a circle which touches two given circles externally is a hyperbola.

4. Show that

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1$$

represents a hyperbola, whose centre is (α, β).

Find its eccentricity and latus rectum.

[*Transfer the origin to* (α, β).]

5. Find the equation of the hyperbola whose foci are $(4, 2)$ and $(8, 2)$ and eccentricity is 2.

6. For the hyperbola $16x^2 - 9y^2 = 144$, find the equation to the diameter which is conjugate to the diameter $x = 2y$.

[*C. U. 1940, '46*]

7. Prove that the lines $x/a + y/b = \lambda$ and $x/a - y/b = 1/\lambda$ where λ is a variable parameter always intersect on a hyperbola.

8. Show that

$$x = \frac{a}{2}\left(t + \frac{1}{t}\right), \quad y = \frac{b}{2}\left(t - \frac{1}{t}\right),$$

where t is a variable parameter, represents a hyperbola.

9. If 2α be the angle between the asymptotes which enclose the hyperbola, then $e = \sec \alpha$.

10. Show that the distance of a focus from an asymptote of a hyperbola is equal to the semi-conjugate axes.

11. Show that the product of the perpendicular distances of any point of a hyperbola from the asymptotes is constant.

12. Show that the pair of tangents drawn from the centre of a hyperbola are its asymptotes.

13. Find the asymptotes of the following hyperbolas

(i) $x^2 - y^2 + 3x - 7y - 3 = 0.$

(ii) $10x^2 - 13xy - 3y^2 - 4x + 23y - b = 0.$

(iii) $axy + bx + cy + d = 0.$

14. Find the equation of a hyperbola whose asymptotes are $2x - y - 3 = 0$ and $3x + y - 7 = 0$ and which passes through $(1, 1)$. [*M. U.*]

15. The asymptotes of a hyperbola are parallel to $2x + 3y = 0$ and $3x + 2y = 0$. Its centre is at $(1, 2)$ and it passes through $(5, 3)$. Find its equation. [*M. U.*]

[*Asymptotes pass through the centre.*]

16. Find the centre of the hyperbola

$$2x^2 + y^2 - 3xy - 5x + 4y + 6 = 0.$$

17. Find the equation of the axes of hyperbola

$$6x^2 - 7xy - 3y^2 = 1.$$

[*Axes are the angle-bisectors of the asymptotes.*]

18. Prove that the polar of any point on an asymptote of a hyperbola is parallel to it.

19. Show that the asymptotes of a hyperbola meet the directrices on the auxiliary circle.

20. Prove that the directrix is the polar of the corresponding focus of a hyperbola.

21. Prove that the line joining the poles of two parallel chords of a hyperbola is a diameter of it.

22. If e_1 and e_2 be the eccentricities of a hyperbola and its conjugate, show that

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1. \qquad \text{[C. U. 1937, '41, '43]}$$

23. Show that the circle described on the line joining the foci of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ as a diameter passes through the foci of its conjugate.

24. Prove that a chord of a hyperbola which touches the conjugate hyperbola is bisected at the point of contact. [*C. U. 1932, '43*]

25. If r_1 and r_2 be two co-directional semi-diameters of a hyperbola and its conjugate, then $r_1^2 + r_2^2 = 0$.

26. If two hyperbolas are conjugate to each other, prove that the polar of a point on one *w. r. t.* the other touches the former.

27. If a pair of conjugate diameters meet the hyperbola and its conjugate respectively in P and D, prove that PD is parallel to one of the asymptotes and is bisected by the other.

28. Find the hyperbolas, conjugate to the following hyperbolas :

(i) $x^2 - y^2 + 3x - 7y - 3 = 0$.

(ii) $10x^2 - 13xy - 3y^2 - 4x + 23y - 6 = 0$.

29. In a rectangular hyperbola show that

(i) $SP.\ S'P = CP^2$.

(ii) $CP = CD$, where CP, CD are two conjugate semi-diameters.

30. Find the co-ordinates of the vertices, and foci and the equation of the directrices of the rectangular hyperbola $xy = c^2$.

31. Any two conjugate diameters of a rectangular hyperbola are equally inclined to the asymptotes.

32. A series of rectangular hyperbolas have the same asymptotes ; show that if two lines form a pair of conjugate diameters *w. r. t.* one of them, they are so *w. r. t.* each of them. [*C. U. 1942*]

33. If the normal to the rectangular hyperbola $xy = c^2$ at 't_1' meets the curve again at 't_2', then

$$t_1^3 t_2 = -1$$ [*C. U. 1945*]

34. There are four points on a rectangular hyperbola. If the chord joining one pair is perpendicular to the chord

joining the other pair, then the same must be true for other pair of chords.

35. Show that $2x^2+3xy-2y^2-5x+5y=0$ represents a rectangular hyperbola. Find its centre and axes.

36. Find the area of the triangle formed by the two asymptotes of the rectangular hyperbola $xy=c^2$ and the normal at the point (x', y') on the hyperbola. [*C. U. 1941*]

37. Show that perpendicular focal chords of a rectangular hyperbola are equal in length.

38. If a line intersects a hyperbola at point P and P' and its asymptotes in Q and Q', show that $PQ=P'Q'$.

[*C. U. 1939, '42*]

[*Use the equation* $xy=c^2$]

39. Prove that the locus of the middle point of the portion (intercepted between two given lines), of a straight line which passes through a fixed point is a hyperbola whose asymptotes are parallel to the given lines. [*C. U. 1936*]

[*Take the two given lines as axes of reference (oblique)*]

40. Find the common tangents of the hyperbolas

(i) $2x^2-3y^2=6, \quad 11x^2-4y^2=44.$

(ii) $b^2x^2-a^2y^2=a^2b^2, \quad b^2y^2-a^2x^2=a^2b^2.$

41. If the polar of a point (α, β) *w. r. t.* the parabola $y^2=4ax$ touches the circle $x^2+y^2=4a^2$, prove that (α, β) lies on the rectangular hyperbola $x^2-y^2=4a^2$. [*C. U. 1933*]

42. Find the locus of the poles of tangents to the auxiliary circle of the hyperbola

$$x^2/a^2-y^2/b^2=1 \text{ w. r. t. the hyperbola.}$$

APPENDIX.

Equation of the ellipse referred to a pair of conjugate diameters as axes of reference.

Let the equation of the ellipse (in the general form) be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad \ldots \quad (1)$$

Since the conjugate diameters are the axes of co-ordinates, every chord parallel to the x-axis is bisected by the y-axis. Hence if we put $y = m$ in (1), the resulting equation

$$ax^2 + 2x(hm + g) + bm^2 + 2fm + c = 0 \qquad \ldots \quad (2)$$

has two roots equal in magnitude but opposite in sign.

$$\therefore \quad hm + g = 0. \qquad \ldots \quad (3)$$

Since this is true for all values of m,

$$\therefore \quad h = 0,\ g = 0.$$

Hence equation (1) reduces to

$$ax^2 + by^2 + 2fy + c = 0. \qquad \ldots \quad (4)$$

Again any chord parallel to y-axis is bisected by x-axis. Hence if we put $x = n$ in (4), the resulting equation

$$by^2 + 2fy + an^2 + c = 0$$

has two roots equal in magnitude but opposite in sign.

$$\therefore \quad f = 0,$$

Thus the equation (1) reduces to

$$ax^2 + by^2 + c = 0$$

$$\text{or} \qquad -\frac{a}{c}x^2 - \frac{b}{c}y^2 = 1. \qquad \ldots \quad (5)$$

Let a', b' be the lengths of the semi-conjugate diameters along the axes of x and y. Putting $y = 0$ and $x = 0$ successively in (5), we get

$$-\frac{a}{c} = \frac{1}{a'^2}, \quad -\frac{b}{c} = \frac{1}{b'^2}.$$

$\therefore$ the reqd. equation is

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1.$$

Cor. When the axes are equi-conjugate diameters, each of length a', the equation of the ellipse is $x^2+y^2=a'^2$, which although of the form of the equation of a circle, does not represent a circle as here the axes are oblique.

Note. Exactly in the same way the equation of the hyperbola referred to its conjugate diameters as axes can be obtained.

Alternative Method.

Let CP, CD be a pair of conjugate diameters and let QVR be any chord parallel to CD, meeting CP in V; then V is the mid-point of QR.

Let α, β, ϕ be the eccentric angles of Q, R, P.

Co-ordinates of V are

$$x=\tfrac{1}{2}a(\cos\alpha+\cos\beta)=a\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta),$$
$$y=\tfrac{1}{2}b(\sin\alpha+\sin\beta)=b\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta).$$

Since V lies on CP, it follows that $\phi=\tfrac{1}{2}(\alpha+\beta)$.

$\therefore$ co-ordinates of P are $a\cos\tfrac{1}{2}(\alpha+\beta)$, $b\sin\tfrac{1}{2}(\alpha+\beta)$

" " D " $-a\sin\tfrac{1}{2}(\alpha+\beta)$, $b\cos\tfrac{1}{2}(\alpha+\beta)$.

Now, $$CV^2=\{a^2\cos^2\tfrac{1}{2}(\alpha+\beta)+b^2\sin^2\tfrac{1}{2}(\alpha+\beta)\}\cos^2\tfrac{1}{2}(\alpha-\beta)$$
$$=CP^2\cos^2\tfrac{1}{2}(\alpha-\beta).$$

Again, $$QV^2=\tfrac{1}{4}QR^2=\tfrac{1}{4}\,[a^2(\cos\alpha-\cos\beta)^2+b^2(\sin\alpha-\sin\beta)^2]$$
$$=\{a^2\sin^2\tfrac{1}{2}(\alpha+\beta)+b^2\cos^2\tfrac{1}{2}(\alpha+\beta)\}\sin^2\tfrac{1}{2}(\alpha-\beta)$$
$$=CD^2\sin^2\tfrac{1}{2}(\alpha-\beta).$$
$$\therefore\ \frac{CV^2}{CP^2}+\frac{QV^2}{CD^2}=1.$$

Now, if CP, CD be taken as axes of co-ordinates (oblique) and $CP=a'$, $CD=b'$ and x, y be the co-ordinates of any point Q on the ellipse *w. r. t.* these axes, then we have

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1.$$

ANSWERS

Examples I

1. (i) $a^2(b-c)+b^2(c-a)+c^2(a-b)$. (ii) 0. (iii) 0. (iv) 0.

2. (i) $a^3+b^3+c^3-3abc=0$.

(ii) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$. **3.** (i) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$. (iv) $\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ a & b & 0 \end{vmatrix}=0$.

4. (i) 3. (ii) 0.

Examples II

2. (4, 5) ; 5. **4.** (i) $\alpha^2+\beta^2=\alpha'^2+\beta'^2$. (ii) $\alpha\alpha'+\beta\beta'=0$.

8. $\left(\dfrac{a^2-ab+2b^2}{a+b}, \dfrac{a^2+3ab-2b^2}{a+b}\right)$, $\left(\dfrac{a^2-3ab-2b^2}{a-b}, \dfrac{a^2+ab+2b^2}{a-b}\right)$.

9. (ii) (10, 4). **12.** (i) 2. (ii) $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

(iii) $2ab \sin \frac{1}{2}(\phi_2-\phi_3) \sin \frac{1}{2}(\phi_3-\phi_1) \sin \frac{1}{2}(\phi_1-\phi_2)$.

(iv) $\frac{1}{2}c^2(t_2-t_3)(t_3-t_1)(t_1-t_2) \div t_1 t_2 t_3$. **16.** 0.

20. (i) $20\frac{1}{2}$. (ii) 3. **22.** (i) $\frac{5}{4}\sqrt{3}$. (ii) $\frac{5}{4}\sqrt{3}.a^2$.

23. (*a*) (i) $r=2a\cos\theta$. (ii) $A\cos\theta+B\sin\theta+C/r=0$.

(iii) $\theta=a$. (iv) $r=a$. (*b*) (i) $x^2+y^2=a^2$. (ii) $x^2+y^2=2ax$.

(iii) $Ax+By=k$. (iv) $(x^2+y^2)^2=a^2(x^2-y^2)$.

(v) $xy=a^2$. (vi) $y^2+4ax=4a^2$.

(vii) $x^3-3xy^2+3x^2y-y^3=5kxy$.

24. (i) $x^2+y^2=a^2$. (ii) $y^2=4ax$. (iii) $3x^2+4y^2-16x-16y+32=0$.

(iv) $ay=c^2$. (v) $r^2-2rr_1\cos(\theta-\theta_1)+r_1^2=a^2$.

Examples III(A)

1. (i) $x\cos\frac{1}{2}(\alpha+\beta)+y\sin\frac{1}{2}(\alpha+\beta)=a\cos\frac{1}{2}(\alpha-\beta)$.

(ii) $y(t_1+t_2)-2x=2at_1t_2$.

(iii) $\dfrac{x}{a}\cos\dfrac{\phi_1+\phi_2}{2}+\dfrac{y}{b}\sin\dfrac{\phi_1+\phi_2}{2}=\cos\dfrac{\phi_1-\phi_2}{2}$.

4. (i) $\frac{1}{4}\pi$. (ii) 0.

7. (i) $\dfrac{a^2}{x'}(x-x')=\dfrac{b^2}{y'}(y-y')$. (ii) $x(y'+f)-y(x'+g)+gy'-fx'=0$.

8. (i) $\left\{\dfrac{a}{m_1 m_2},\ a\left(\dfrac{1}{m_1}+\dfrac{1}{m_2}\right)\right\}$. (ii) $\left\{a\dfrac{\cos\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')},\ b\dfrac{\sin\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')}\right\}$.

12. (i) $11x-2y-9=0$. **13.** $2x-y-3=0$.

14. (i) $119x+102y=125$. (ii) $8y=7x+9$. (iii) $x+y-9=0$.

16. (ii) $k=5$; $(-2, 3)$. **23.** $bc+ca+ab$.

28. $(\frac{1}{2}a, \frac{1}{2}b')$, $(\frac{1}{2}a', \frac{1}{2}b)$, $\left\{-\dfrac{aa'(b-b')}{2(ab'-a'b)},\ \dfrac{bb'(a-a')}{2(ab'-a'b)}\right\}$.

29. $5x+69y-28=0$. $37x+29y-112=0$.

31. $3x+y=8$; $x-3y=6$. **32.** (a) (i) $(\frac{66}{25}, \frac{9}{25})$.

(ii) $\left\{-a,\ a\left(\dfrac{1}{m_1}+\dfrac{1}{m_2}+\dfrac{1}{m_3}+\dfrac{1}{m_4 m_2 m_3}\right)\right\}$. (b) $(\frac{6}{5}, \frac{9}{5})$.

33. (i) $\frac{3}{2}$. (ii) $\frac{1}{2}(c_1-c_2)^2 \div (m_1-m_2)$.

(ii) $\dfrac{1}{2}\left\{\dfrac{(c_2-c_3)^2}{m_2-m_3}+\dfrac{(c_3-c_1)^2}{m_3-m_1}+\dfrac{(c_1-c_2)^2}{m_1-m_2}\right\}$.

Examples III(B)

1. $(\frac{1}{2}, 1)$. **5.** 1. **7.** $(0, 0)$. **10.** $2ax+2by+c+c'=0$.

11. 3. **12.** $(\frac{2}{5}, \frac{9}{5})$; 2. **13.** $3x+y-8=0$.

15. $7x-4y+16=0$, $8x+14y+9=0$. **16.** $x+y=0$, $x-y=0$.

18. $2x+y-3=0$, $x+3y-4=0$, $x-y=0$. **20.** $(\frac{48}{49}, \frac{111}{49})$; $(1, 1)$.

22. A right line. **35.** $\{-\frac{1}{2}k \operatorname{cosec}\frac{1}{2}\alpha \operatorname{cosec}\frac{1}{2}\beta,\ \frac{1}{2}(\alpha+\beta)\}$.

40. $x^2+y^2-ax-by=0$.

Examples IV

1. (i) $2x'+3y'=0$. (ii) $x'^2+y'^2=37$. (ii) $3y'^2=7x'$.

(ii) $5x'^2-4y'^2=16$. **2.** $\alpha=3$, $\beta=-5$; $A=\frac{1}{3}$, $B=\frac{1}{15}$.

3. $(1, -1)$. **4.** $(2, 3)$; $\lambda=-4$, $\mu=-3$, $v=5$. **9.** $\frac{1}{6}\pi$; 2.

7. $-45°$, $\frac{1}{2}a^2$. **8.** $2xy+a^2=0$. **10.** $45°$.

11. $h=-\frac{3}{2}$, $k=-\frac{5}{2}$; the equation reduces ts $12x'^2-10x'y'+2y'^2=0$.

Examples V

1. (i) $\tan^{-1} 8$. (ii) a. (iii) $\frac{1}{2}\pi$.

2. (i) $x^2-y^2=0$. (ii) $(a^2-b^2)(x^2-y^2)=2c^2xy$.

10. $3x^2-7xy-5y^2=0$. **11.** $(aa'-bb')^2+4(ah'+b'h)(a'h+bh')=0$.

13. (i) 36. **16.** (i) $(-1, 1)$; $\tan^{-1}\frac{1}{7}$: $(1, 3)$, $90°$; $(4, -7)$; $\tan^{-1}\frac{1}{3}$.

(iv) $(-1, 2)$; $\tan^{-1}\frac{3}{2}$.

17. (i) $-\frac{1}{2}, \frac{1}{5}$; (ii) ± 1, **18.** (iii) $\frac{2}{\sqrt{5}}$.

25. $7x^2 - 22xy - 7y^2 - 86x + 38y + 167 = 0$. **29.** 9, or -4.

Examples VI(A)

2. 4. **6**, (i) $x^2 + y^2 - 10x - 10y + 25 = 0$.
(ii) $x^2 + y^2 - 15x - 12y + 36 = 0$. **7.** $x^2 + y^2 - 7x - 13y - 52 = 0$.

8. $x^2 + y^2 - 8x + 12y - 49 = 0$. **9.** (i) Outside the circle. (ii) No.

10. No. **11.** $2\sqrt{g^2 - c}$ (on x-axis), $2\sqrt{f^2 - c}$ (on y-axis).

12. $(-1, -1)$. **13.** $(x-h)\cos\alpha + (y-k)\sin\alpha = a$.

18. $(a, 0), (0, a)$. **19.** (i) $4x + 3y + 5 = 0$; $4x + 3y - 25 = 0$.
(ii) $3x - 4y - 10 = 0$; $3x - 4y + 20 = 0$. **20.** $(3, 5)(-1, -11)$.

25. $(\frac{1}{2}, -\frac{3}{2})$. **29.** 8. **30.** (i) st. line. (ii) circle.

34. $x^2 + y^2 + x(2g + cl) + y(2f + mc) = 0$.

35. $(l^2 + m^2)(x^2 + y^2) - 2lx - 2my - \{a^2(l^2 + m^2) - 2\} = 0$.

Examples VI(B)

3. Circle.

5. If (h, k) be the fixed point and $x^2 + y^2 = a^2$ be the given circle, then the centre of the locus is $(\frac{1}{2}h, \frac{1}{2}k)$ and its radius is $\frac{1}{2}a$.

7. $(x^2 + y^2)^2 + 2(gx + fy)(x^2 + y^2) - (gy - fx)^2 + e(x^2 + y^2) = 0$.

8. $2(x^2 + y^2) + 2gx + 2fy + c = 0$.

10. (i) $\left(-\frac{ln}{l^2 + m^2}, -\frac{mn}{l^2 + m^2}\right)$. (ii) $3y - 2x = 13$.

11. $x^2 + y^2 = hx + ky$. **12.** $x^2 + y^2 = \frac{1}{2}a^2$.

15. $6x^2 - 6y^2 - 5xy + 13x + 65y - 169 = 0$. **16.** $3x^2 - 10xy + 3y^2 = 0$.

19. (i) $\left(\sqrt{a^2 + b^2}, \tan^{-1}\frac{b}{a}\right)$; $\sqrt{a^2 + b^2}$. (ii) $(5, 30°)$; 5.

20. $(\sqrt{2}a, 45°)$, $(-\sqrt{2}a, 135°)$.

24. $r^2 - r\{r_1\cos(\theta - \theta_1) + r_2\cos(\theta - \theta_2)\} + r_1 r_2\cos(\theta_1 - \theta_2) = 0$.

Exampels VII

1. $\frac{1}{2}$. **3.** $x^2 + y^2 + 6x - 3y = 0$. **5.** $ax - by = 0$.

6. (i) $(-\frac{4}{3}, -\frac{4}{3})$. (ii) $(-2, -1)$; 2.

8. $x^2 + y^2 - 9x - 4y + 12 = 0$.

10. $2(x^2 + y^2) + 2x + 6y + 1 = 0$. **11.** (i) 10.

(ii) $\sqrt{4r^2-2(p-q)^2}$. 12. $(1, 1), (-2, -2)$.

16. (i) $x^2+y^2-8x-8y+16=0$. (ii) $x^2+y^2-16x-18y-4=0$.
(iii) $x^2+y^2-cx-by+a^2=0$.

17. $2g_2(g_1-g_2)+2f_2(f_1-f_2)=c_1-c_2$.

18. (i) $24x+7y=125$; $4x-3y=25$; $7x-24y=250$; $3x+4y=50$.
(ii) $x=1$, $y=2$, $4x-3y=10$, $3x+4y=5$.

21. $3x+y-4=0$. 22. (i) $(5, 4)$ $(-5, -4)$. (ii) $(2, 0)$, $(0, -2)$.

Examples VIII(A).

1. (i) $x^2-2xy+y^2-18x-10y+45=0$.
(ii) $16x^2+24xy+9y^2-256x-142y+849=0$.
(iii) $x^2+y^2+2xy-4x+4y-4=0$.
(iv) $x^2-2xy+y^2+24x+8y+16=0$.
(v) $5\{(x+6)^2+(y+6)^2\}=(x+2y-22)^2$. (vi) $y^2=4(b-a)(x-a)$.

2. $(0, 0)$; $4x-3y=0$. 3. (i) Vertex $(\frac{7}{4}, 1)$; focus $(3, 1)$; axis $y=1$; directrix $2x-1=0$; latus rectum$=5$.
(ii) vertex $(-\frac{3}{2}, -\frac{61}{40})$; focus $(-\frac{3}{2}, -\frac{41}{40})$; axis $x+\frac{3}{2}=0$; directrix $y+\frac{81}{40}=0$; latus rectum$=2$. 4. $y=2x^2-7x+2$.

6. inside ; upon ; outside. Read $y^2=16x$ for $y^2=4ax$.

7. Vertex $\left(\dfrac{u^2\sin 2a}{2g}, \dfrac{u^2\sin^2 a}{2g}\right)$; latus rectum$=\dfrac{2u^2\cos^2 a}{g}$.

12. $x-y+a=0$, $x+y+a=0$; $x+y=3a$; $x-y=3a$.

13. (i) $3y=2x+9$; (ii) $y=2x+1$.

14. $(\frac{1}{3}a, \frac{2}{3}\sqrt{3a})$. 16. $\tan^{-1}\dfrac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}}+b^{\frac{2}{3}})}$.

26. $y^2(x+2a)+4a^3=0$. 33. (i) $a^{\frac{1}{3}}x+b^{\frac{1}{3}}y+a^{\frac{2}{3}}b^{\frac{2}{3}}=0$.
(ii) $y=\pm(x+2a.)$

Examples VIII(B)

1. $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$. 5. $y^2-ky=2a(x-h)$.

6. $y^2(y^2-2ax+4a^2)+8a^4=2$. 7. $y^2=2a(x-4a)$.

8. $y^2=2a(x-a)$. 9. $(\frac{5}{4}, 3)$.

11. $y^2+2x^2+4xy+12x+4y+2=0$.

12. $\dfrac{\sqrt{y_1^2+4a^2}\cdot\sqrt{y_1^2-4ax_1}}{a}$. 13. $\dfrac{(y_1^2-4ax_1)^{\frac{3}{2}}}{2a}$.

14. $(y-5)^2=8(x-3)$.
15. $\{a(t_1^2+t_2^2+t_1t_2)+2a,\ -at_1t_2(t_1+t_2)\}$.
21. (i) $y^2-4ax=3(x+a)^2$. (ii) $y^2-4ax=(x+a)^2$.
(iii) $3(y^2-4ax)=(x+a)^2$ 22. $y^2-2ax=kx^2$.
23. (i) $x^2=k^2\{y^2+(x-a)^2\}$. (ii) $y^2=4ax+a^2k^2$.
26. $(2, -4)$, $(18, -12)$, $(32, 16)$.
27. $y=x-6$; $y=2x-24$; $y+3x=66$. 30. $\frac{16}{3}a$.

Examples IX(A)

1. (i) $5x^2+5y^2+2xy+10x-22y+39=0$.
(ii) $7x^2+7y^2+2xy+10x-10y+7=0$.
2. $\left(\frac{1}{3}, \frac{1}{3}\right)$. 3. $\frac{4x^2}{81}+\frac{4y^2}{45}=1$.
4. (i) $(-2, 3)$, $(4, 3)$. (ii) $e=\frac{1}{2}$; latus rectum $=3$.
7. upon, inside, outside. 8. (i) $\frac{1}{4}\pi$, $\frac{3}{4}\pi$. (ii) 45°, 135°, 225°, 315°.
9. 5. 11. $x^2/5^2+y^2/4^2=1$.
14. (i) $y=\pm x\pm 5$. (ii) $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}},\ \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$.
15. (i) $(1, -\frac{1}{2})$, $(-1, \frac{1}{2})$; (ii) $(2, 5)$, $(-2, -5)$.
27. (i) (a^2l, b^2m). (ii) $(a^2\cos\alpha/p,\ b^2\sin\alpha/p)$.
(iii) $a\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)},\ b\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}$.
30. (i) $a^2x^2+b^2y^2=a^4$. (ii) $x^2/a^4+y^2/b^4=1/a^2$.
34. (i) $\frac{x}{a}\sin\delta-\frac{y}{b}\cos\delta=0$. (ii) $x^2/a^2+y^2/b^2=\sec^2\delta$.
37. $hx/a^2+ky/b^2=1$. 38. $a^6/x^2+b^6/y^2=(a^2-b^2)^2$.
41. $x^2+y^2=13$. 44. $a^4l^2+b^4m^2=a^2+b^2$.

Examples IX(B)

1. $\left(\frac{a^2l}{a^2l^2+b^2m^2},\ \frac{b^2m}{a^2l^2+b^2m^2}\right)$.
2. (i) $b^2x(x-\alpha)+a^2y(y-\beta)=0$.
(ii) $(a^2+b^2)(b^2x^2+a^2y^2)^2=a^2b^2(b^4x^2+a^4y^2)$.
(iii) $(b^6x^2+a^6y^2)(b^2x^2+a^2y^2)^2=a^4b^4x^2y^2(a^2-b^2)^2$.
(iv) $a^4b^4(x^2+y^2)=(a^2+b^2)(b^2x^2+a^2y^2)^2$.
3. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2}$. 14. (i) $x^2/a^2+y^2/b^2=1/2$.
(ii) $2(x^2+y^2)^2=a^2x^2+b^2y^2$. (iii) $x^2/a^2+y^2/b^2=2$.

18. $y+2x=0,\ x-3y=0\ ;\ y-2x=0,\ x+3y=0.$

21. $x^2-8xy+6y^2+8x+8y-24=0\ ;\ \tan^{-1}(\frac{2}{7}\sqrt{10}).$

22. $x-4y-13=0,\ 4x-5y+25=0.$ Read $9x^2+25y^2=225$ *for* $9x^2+25x^2=225.$ 23. (i) $k(x^2-a^2)=2xy.$ (ii) $\lambda(y^2-b^2)=2xy.$

24. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x^2+y^2}{a^2-b^2}.$

25. $x\sqrt{\beta^2-b^2}\pm y\sqrt{a^2-\alpha^2}\pm\sqrt{(a^2\beta^2-b^2\alpha^2)}=0.$

30. (i) $\tan^{-1}\frac{1}{3}$; (ii) $\frac{1}{6}\pi$; (iii) $\frac{1}{4}\pi.$

Examples X

1. (i) $x^2+y^2+4xy-32x-32y+154=0.$
   (ii) $4x^2+11y^2-24xy-50x-160y-225=0.$ 2. $16x^2-9y^2=144.$

4. $e=\frac{\sqrt{a^2+b^2}}{a}$, latus rectum $=2b^2/a.$

5. $\frac{(x-6)^2}{1}-\frac{(y-2)^2}{3}=1.$ 6. $9y=32x.$

13. (i) $x+y+5=0,\ x-y-2=0.$ (ii) $5x+y-7=0,\ 2x-3y+2=0.$
   (iii) $ax+c=0,\ ay+b=0.$ 14. $6x^2-xy-y^2-23x+4y+15=0.$

15. $(2x+3y-8)(3x+2y-7)-154=0.$ 16. $(2, 1).$

17. $7(x^2-y^2)+18xy=0.$ 28. (i) $x^2-y^2+3x-7y-17=0.$
   (ii) $10x^2-13xy-3y^2-4x+23y-22=0.$

30. Vertices $(\pm c,\ \pm c).$ Foci $(\pm c\sqrt{2},\ \pm c\sqrt{2}).$
   Directrices $x+y=\pm c\sqrt{2}.$

35. Centre $(\frac{1}{5}, \frac{7}{5})$; axes : $3(x^2-y^2)=8xy$ referred to centre as origin.

36. $\frac{1}{2}\left(\frac{x'^2-y'^2}{c}\right)^2.$ 40. (i) $y=3x+5,\ y=3x-5\ ;\ y=-3x+5,\ y=-3x-5.$ (ii) $y=\pm x\pm\sqrt{a^2-b^2}.$ 42. $b^4x^2+a^4y^2=a^2b^4.$